1. (A1) (4 points) For this problem: let $\mathbf{v}$ and $\mathbf{w}$ be 3D vectors so $\|\mathbf{v}\| = 3$ and $\|\mathbf{w}\| = 1$.

(a) Find the angle between the vectors $\mathbf{x} = \mathbf{v} + 3\mathbf{w}$ and $\mathbf{y} = \mathbf{v} - 3\mathbf{w}$ using **only** the information at the top of the page. Hint: there is enough information. You do **not** need to find $\|\mathbf{x}\|$ or $\|\mathbf{y}\|$ to answer this problem.

**Solution:**

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{v} + 3\mathbf{w}) \cdot (\mathbf{v} - 3\mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 9\ \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - 9\|\mathbf{w}\|^2 = 9 - 9(1) = 0$$

The vectors are orthogonal.

(b) Let $\mathbf{v} = \langle 2, 2, 1 \rangle$ and $\mathbf{w} = \langle 0, 0, 1 \rangle$. Find the orthogonal projection $\text{proj}_{\mathbf{v}}(\mathbf{w})$ of $\mathbf{w}$ onto $\mathbf{v}$.

**Solution:**

$$\text{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{1}{9} \langle 2, 2, 1 \rangle$$

(c) Find the area of the parallelogram formed by $\mathbf{v} = \langle 2, 2, 1 \rangle$ and $\mathbf{w} = \langle 0, 0, 1 \rangle$.

**Solution:** The area is:

$$\|\mathbf{v} \times \mathbf{w}\| = \|\langle 2, -2, 0 \rangle\| = \sqrt{8}$$

(d) Find a **unit** vector orthogonal to both $\mathbf{v} = \langle 2, 2, 1 \rangle$ and $\mathbf{w} = \langle 0, 0, 1 \rangle$.

Hint: these are the same vectors from c.

**Solution:** We scale $\mathbf{v} \times \mathbf{w}$ to be a unit vector:

$$\frac{1}{\sqrt{8}} \langle 2, -2, 0 \rangle$$

2. (A2) (4 points) Let $\mathcal{P}$ the plane that contains all points on the line with parametrization:

$$\ell : \mathbf{x} = \langle 2 + 3t, 5 + t, 4t \rangle$$

and is parallel to (but does not have to contain) the line $\ell'$ thru the points $Q(4, 1, 1)$ and $R(6, 3, 4)$. Note: parallel means a direction vector for this line is also one of the direction vectors for the plane.

(a) Find an equation equivalent to one of form $ax + by + cz = d$ for $\mathcal{P}$. Note: the plane contains $\ell$, not $\ell'$.

**Solution:** A point on the plane is a point on the line $\ell$ like $P(2, 5, 0)$. A normal vector to the plane is given by a cross product of direction vectors for the lines:

$$\mathbf{n} = \underbrace{\langle 3, 1, 4 \rangle}_{\ell \text{ dirn}} \times \underbrace{\langle 2, 2, 3 \rangle}_{\ell' \text{ dirn: } \mathbf{QR}} = \langle -5, -1, 4 \rangle$$

An equation for the plane is:

$$-5(x - 2) - 1(y - 5) + 4(z - 0) = 0$$

(b) Find the distance between the lines $\ell$ and $\ell'$.

**Solution:** We use the distance formula with $\mathbf{n} = \langle -5, -1, 4 \rangle$ and points $P(2, 5, 0)$ on $\ell$ and $Q(4, 1, 1)$ on $\ell'$. The distance is:

$$\left| \frac{\mathbf{n} \cdot (\mathbf{q} - \mathbf{p})}{\|\mathbf{n}\|} \right| = \left| \frac{\langle -5, -1, 4 \rangle \cdot \langle 2, -4, 1 \rangle}{\sqrt{42}} \right| = \frac{2}{\sqrt{42}}$$

3. (A3) (4 points) This problem has multiple parts.

(a) The curves with the parametrizations given below intersect at a **single** point. Find it.

$$\mathcal{C}_1 : \mathbf{r}_1(t) = \langle t^2 + 2, 4, t \rangle$$

$$\mathcal{C}_2 : \mathbf{r}_2(t) = \langle t, t - 2, t - 8 \rangle$$

Hint: is it a good idea to use the same parameter (i.e. $t$) for both curves? Note: your answer should be a point.

**Solution:** We rename the parameter for the first curve as $s$ and set up:

$$\begin{aligned} s^2 + 2 &= t \\ 4 &= t - 2 \\ s &= t - 8 \end{aligned}$$

Sub the first equation into the second gives $s^2 = 4$ and putting this into the first equation gives $t = 6$. Putting this into the last equation gives $s = -2$.

The point in question has position vector $\mathbf{r}_1(-2) = \mathbf{r}_2(6) = \langle 6, 4, -2 \rangle$ and so is $P(6, 4, -2)$.

(b) Find the angle between the curves (from part a) at the point of intersection.

You may leave your answer unsimplified.

**Solution:** The directions of the curves at the point of intersection are $\mathbf{r}_1'(-2) = \langle -4, 0, 1 \rangle$ and $\mathbf{r}_2'(6) = \langle 1, 1, 1 \rangle$. The angle $\theta$ is found by setting up:

$$\begin{aligned} &\langle -4, 0, 1 \rangle \cdot \langle 1, 1, 1 \rangle = \|\langle -4, 0, 1 \rangle\| \|\langle 1, 1, 1 \rangle\| \cos\theta \\ &-3 = \sqrt{17}\sqrt{3}\cos\theta \\ &\cos^{-1}\left(-\tfrac{3}{\sqrt{51}}\right) = \theta \end{aligned}$$

(c) Name the surface described by the equation $-4x^2 + y^2 - z^2 = -16$. No work required.

paraboloid saddle ellipsoid/sphere 1-sheeted hyperboloid 2-sheeted hyperboloid cone

**Solution:** Divide by $-16$ to get:

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{4}\right)^2 + \left(\frac{z}{4}\right)^2 = 1$$

It is a hyperboloid of 1–sheet.

4. (A4) (4 points) For this problem consider the function $f(x,y) = e^{xy+x}$.

(a) Find all first–order **and** second–order partial derivatives of $f$. Hint: at least one of the second–order partial derivatives will require the product rule.

**Solution:** We get $f_x(x,y) = (y+1)e^{xy+x}$ and $f_y(x,y) = xe^{xy+x}$. Then:

$$f_{xx}(x,y) = (y+1)^2 e^{xy+x}$$
$$f_{xy}(x,y) = f_{yx}(x,y) = e^{xy+x} + x(y+1)e^{xy+x}$$
$$f_{yy}(x,y) = x^2 e^{xy+x}$$

(b) Find an equation of the tangent plane to the graph of $f$ at the point $(0,0,\star)$ on the graph. Note: while I don't tell you what $\star$ is, you could figure it out.

**Solution:** We find $f_x(0,0) = 1$ and $f_y(0,0) = 0$ and $f(0,0) = 1$. So the equation is:

$$z - 1 = 1(x-0) + 0(y-0) \implies z = 1 + x$$

(c) Use your tangent plane from part b to estimate $f(-0.1, 0.1)$ with a linear approximation.

**Solution:** We plug $x = -0.1$ and $y = 0.1$ into the equation for the tangent plane to get:

$$z = 1 + (-0.1) = 0.9$$

So $f(-0.1, 0.1) \approx 0.9$

5. (A5) (4 points) For this problem let $f(x,y) = \sin(2x^2y)$.

(a) (i) Find the direction in which the rate of change of $f$ is most **negative** at the point $P(1,0)$.

Note: the question says most negative, not positive. Hint: your answer should be a (not necessarily unit) vector.

(ii) Find this most negative rate of change of $f$ at $P$. Hint: your answer should be a (hopefully negative) number.

**Solution:** We find:

$$\nabla f(x,y) = \langle 4xy\cos(2x^2y), 2x^2\cos(x^2y)\rangle \implies \nabla f(1,0) = \langle 0, 2\rangle$$

The direction of greatest decrease is therefore:

$$-\nabla f(1,0) = \langle 0, -2\rangle$$

This most negative rate of change is:

$$-\|\nabla f(1,0)\| = -2$$

(b) Continue to let $f(x,y) = \sin(2x^2y)$ and now let $x = s - t$ and $y = s^2t$. Find $\dfrac{\partial f}{\partial t}$ using the chain rule.

**Solution:** We find:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 4xy\cos(2x^2y)\cdot(-1) + x^2\cos(x^2y)\cdot s^2$$

(c) Consider the plane $\mathcal{P}$ defined by $x + y + z = 1$ and the surface $\mathcal{S}$ defined by $-x^2 + y^2 + z^2 = 1$.

Which of the following planes (i, ii, neither, or both) is parallel to the plane $\mathcal{P}$? Note: work required.

Hint: planes are parallel when their normal vectors are parallel.

(i) the tangent plane to $\mathcal{S}$ at $Q(1,1,1)$ (ii) the tangent plane to $\mathcal{S}$ at $R(-1,1,1)$

**Solution:** The plane $x + y + z = 1$ has normal $\mathbf{n} = \langle 1, 1, 1\rangle$.

The tangent plane at $Q(1,1,1)$ has normal $\nabla f(1,1,1) = \langle -2, 2, 2\rangle$ which is not parallel to $\mathbf{n}$. So the tangent plane at $Q$ is not parallel to the plane $\mathcal{P}$.

The tangent plane at $R(-1,1,1)$ has normal $\nabla f(-1,1,1) = \langle 2, 2, 2\rangle$ which is parallel to $\mathbf{n}$ because it equals $2\mathbf{n}$. So the tangent plane at $R$ is parallel to the plane $\mathcal{P}$.

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