1. (B1) (4 points) For this problem let $f(x,y) = xy + \dfrac{1}{x} - 4y^2$.

(a) Find all critical points of f and classify them (local min, local max, or saddle). Hint: there is only one.

Solution: We find critical points by setting the gradient equal to 0:

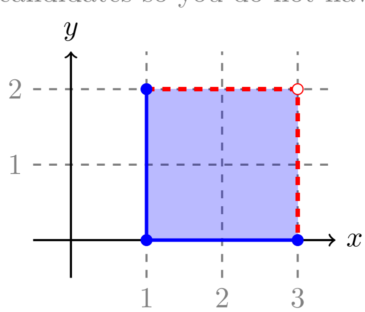
$$y - \tfrac{1}{x^2} = 0 \implies y = \tfrac{1}{x^2}$$
$$x - 8y = 0 \implies x = 8y$$

We substitute the first equation into the second to get $x = \frac{8}{x^2} \implies x^3 = 8 \implies x = 2$. The critical point is $(2, \frac{1}{4})$. Next we find the Hessian:

$$Hf(x,y) = \begin{pmatrix} \frac{2}{x^3} & 1 \\ 1 & -8 \end{pmatrix} \implies Hf\left(2, \frac{1}{4}\right) = \begin{pmatrix} \frac{1}{4} & 1 \\ 1 & -8 \end{pmatrix}$$

The determinant is -3. This is a saddle point.

(b) Find a **complete but finite** list of candidates at which f **could** have an absolute extremum on the square $[1,3] \times [0,2]$ below, assuming that an absolute extremum does **not** occur on the right edge **nor** top edge **nor** the top-right corner (all in dashed red). Note: I give you this assumption to save you time! Also: I only ask you for candidates so you do not have to waste time plugging into f.



Note: anything that is non–dashed should be explicitly addressed.

Solution: One candidate is the critical point $(2, \frac{1}{4})$ from part a because it is feasible…though you could argue that it is not a candidate because it is a saddle point in the interior.

Three more candidates are the corners $(1,0)$, $(3,0)$, and $(1,2)$.

Next we parametrize the left edge using: $(1,y)$ with $0 < y < 2$. We find:

$$f(1,y) = y + 1 - 4y^2$$

which has derivative $1 - 8y \implies y = \frac{1}{8}$. This is feasible and gives us candidate $(1, \frac{1}{8})$.

Lastly we parametrize the bottom edge use: $(x,0)$ with $1 < x < 3$ which gives us $f(x,0) = \frac{1}{x}$ which has no critical points, so yields no candidates.

A complete list is: $(2, \frac{1}{4})$, $(1, \frac{1}{8})$, $(1,0)$, $(3,0)$, $(1,2)$.

your complete list of (x,y) absolute extremizer candidates:

2. (B2) (4 points) This problem has two parts.

(a) Use Lagrange multipliers to find radius of the largest sphere centered at the origin that can be inscribed in the ellipsoid:

$$4x^2 + 3y^2 + 6z^2 = 12$$

You do **not** have to check the degenerate case. Note: only the Lagrange multipliers method will receive credit. Hint: consider the distance–squared to origin function. Hint: when solving the system, break into three cases, each based on: is this variable nonzero? There is **no** candidate where all three coordinates are nonzero.

Solution: We minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g = 12$ where $g(x, y, z) = 4x^2 + 3y^2 + 6z^2$. The Lagrange system is:

$$\begin{cases} 8x = 2x\lambda \\ 6y = 6y\lambda \\ 12z = 2z\lambda \\ 4x^2 + 3y^2 + 6z^2 = 12 \end{cases}$$

If $x \neq 0$ then the first equation yields $\lambda = 4$ which, when substituted into the 2nd and 3rd equations, yields $y = z = 0$. Substituting into the constraint gives $4x^2 = 12 \implies x = \pm\sqrt{3}$. So we have found candidates:

$$(\pm\sqrt{3}, 0, 0)$$

Similarly if $y \neq 0$ we would find $x = z = 0$ and so $3y^2 = 12 \implies y = \pm 2$ yielding candidates:

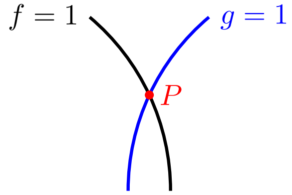
$$(0, \pm 2, 0)$$

And lastly if $z \neq 0$ we would find $x = y = 0$ and so $6z^2 = 12 \implies z^2 = 2 \implies z = \pm\sqrt{2}$ yielding candidates:

$$(0, 0, \pm\sqrt{2})$$

The candidates that are closest to the origin are $(0, 0, \pm\sqrt{2})$. Their distance from the origin is $\sqrt{2}$.
So the radius of the largest sphere that can be inscribed in the ellipsoid is $r = \sqrt{2}$.

(b) The point P is on both the level curves $f = 1$ and $g = 1$. Is it possible that P is an extremizer of f subject to the constraint $g = 1$? Briefly explain. Note: you may assume the gradients of f and g are nonzero at P. Note: an answer without correct explanation receives no credit.

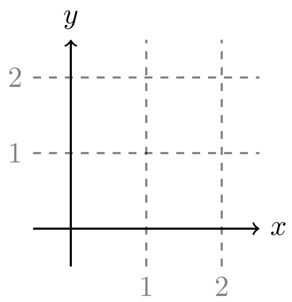


Solution: No. The gradients ∇f and ∇g are not parallel. This violates the Lagrange multipliers condition.

3. (B3) (4 points) This problem has multiple parts.

(a) Find the integral. Hint: the given order of integration is no good.

$$\int_0^1 \int_{2x}^2 \frac{\cos(\pi y)}{y}\ dydx$$



Solution: The region of integration is the triangle with sides $y = 2x$ and $y = 2$ and $x = 0$. Sketch it!

In the other order of integration: is between $x = 0$ and $x = \frac{y}{2}$ from $0 \leq y \leq 2$.

So the integral can be rewritten as:

$$\int_0^2 \int_0^{\frac{y}{2}} \frac{\cos(\pi y)}{y}\ dxdy = \int_0^2 \frac{\cos(\pi y)}{2}\ dy = \frac{\sin(2\pi)}{2\pi} - \frac{\sin(0)}{2\pi} = 0$$

(b) Set up **but do not compute** a double integral in order $dydx$ or $dxdy$ (your choice) that equals the volume of the tetrahedron bounded by the plane $3x + 3y + 5z = 15$ and the coordinate planes ($x = 0$, $y = 0$, $z = 0$).

Solution: The top of the tetrahedron is:

$$3x + 3y + 5z = 15 \implies z = 3 - \frac{3x}{5} - \frac{3y}{5}$$

The shadow of the tetrahedron in the xy–plane has sides $x = 0$, $y = 0$, and:

$$3x + 3y = 15 \implies y = 5 - x$$

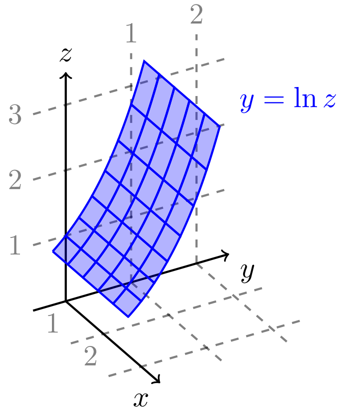
The double–integral that equals the volume of the tetrahedron is therefore:

$$\int_0^5 \int_0^{5-x} 3 - \frac{3x}{5} - \frac{3y}{5}\ dydx$$

4. (B4) For this problem let E be the combined region of integration of:

$$\int_0^1 \int_1^e \int_{\ln z}^1 f(x,y,z)\; dydzdx + \int_0^1 \int_0^1 \int_0^1 f(x,y,z)\; dydzdx.$$

(a) Rewrite the sum of iterated integrals as a **single** integral in the order $dzdydx$. Hint: $a = \ln b \iff e^a = b$.



Solution: Knowing that it is possible to write this in the order $dzdydx$ we first identify the shadow in the xy-plane. According to the bounds given: $0 \leq x \leq 1$, and the smallest value of y is 0, and the largest value of y is 1. Therefore the shadow in the xy-plane is $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Next we have to find the bounds on z. The first summand gives bounds involving z as:

$$1 \leq z \leq e$$

and:

$$\ln z \leq y \leq 1 \implies z \leq e^y \leq 1$$

and the second summand gives bounds:

$$0 \leq z \leq e$$

Using the most restrictive bounds: we see that that $0 \leq z \leq e^y$ and so the integral is:

$$\int_0^1 \int_0^1 \int_0^{e^y} f(x,y,z)\; dzdydx$$

(b) Let $\delta(x,y,z) = 8xz$ be the density of mass at point (x,y,z) in the region E. Find the mass of E. Hint: if you could not solve part (a) you may need that $\int t \ln t\; dt = \frac{t^2 \ln t}{2} - \frac{t^2}{4} + C$.

Solution: We compute:

$$\int_0^1 \int_0^1 \int_0^{e^y} 8xz\; dzdydx = \int_0^1 \int_0^1 4xe^{2y}\; dydx = \int_0^1 2x(e^2 - 1)\; dx = e^2 - 1$$

5. (B5) (4 points) This problem has multiple parts.

(a) Let E be the region bounded below by the upwards paraboloid $z = x^2 + y^2 - 4$ and bounded above by the flipped cone $z = 2 - \sqrt{x^2 + y^2}$ and find:

$$\iiint_E \frac{1}{\sqrt{x^2+y^2}}\, dV$$

Hint: first write the surfaces in cylindrical coordinates. The surfaces intersect in a circle, so find its radius: you may need to factor to solve (and maybe ignore one non–relevant solution).

Solution: In cylindrical coordinates the surfaces are $z = r^2 - 4$ and $z = 2 - r$. They intersect when:

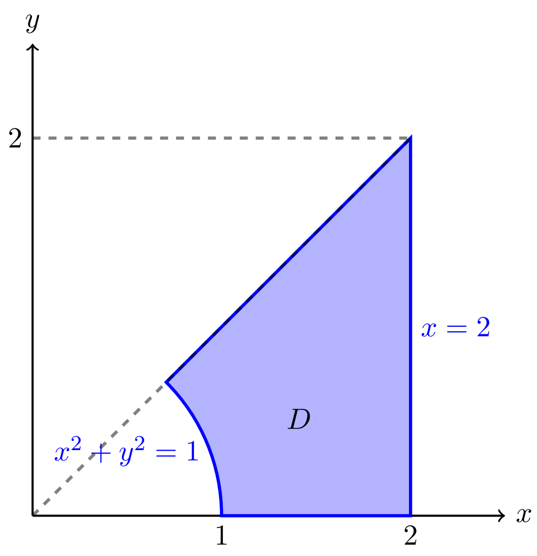
$$r^2 - 4 = 2 - r \implies r^2 + r - 6 = 0 \implies (r+3)(r-2) = 0$$

and so $r = 2$ (we ignore the negative solution). Therefore our triple integral is:

$$\int_0^{2\pi}\int_0^2\int_{r^2-4}^{2-r} \frac{1}{r}\cdot r dz dr d\theta = \int_0^{2\pi}\int_0^2 (2-r)-(r^2-4)\, dr d\theta = \int_0^{2\pi}\int_0^2 6-r-r^2\, dr d\theta = 2\pi\left[12 - 2 - \frac{8}{3}\right] = \frac{44\pi}{3}$$

(b) Set up **but do not compute** the integral $\iint_D x\, dA$ using polar coordinates in order $dr d\theta$.

Note: make sure **both** the bounds **and** the integrand (function being integrated) involve only r and θ.



Solution: The range of angles is $0 \le \theta \le \frac{\pi}{4}$. The circle edge is given by $r = 1$. The edge $x = 2$ can be written as $r\cos\theta = 2 \implies r = 2\sec\theta$. So we have $1 \le r \le 2\sec\theta$. This allows us to write:

$$\iint_D x\, dA = \int_0^{\pi/4}\int_1^{2\sec\theta} r\cos\theta \cdot r dr d\theta$$

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