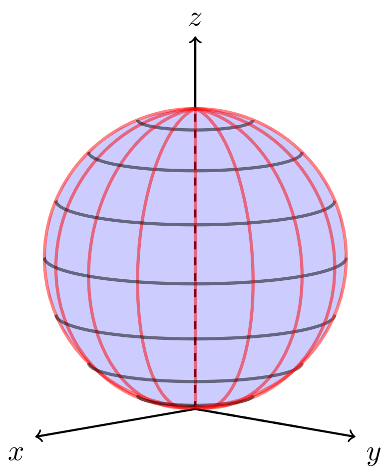
1. (C1) (4 points) This question has multiple parts.

   (a) Let B be the shifted–up solid ball given by $x^2 + y^2 + (z-2)^2 \leq 4$. Find the bounds in order $d\rho d\phi d\theta$ if B is the region of integration.



$\leq \rho \leq$
$\leq \phi \leq$
$\leq \theta \leq$

Solution: We rewrite the inequalities as:

$$x^2 + y^2 + z^2 - 4z + 4 \leq 4 \implies x^2 + y^2 + z^2 \leq 4z \implies \rho^2 \leq 4\rho\cos\phi \implies \rho \leq 4\cos\phi$$

So the bounds on ρ are $0 \leq \rho \leq 4\cos\phi$.

Using the picture we have $0 \leq \phi \leq \frac{\pi}{2}$ (the ball is above the xy-plane) and $0 \leq \theta \leq 2\pi$.

   (b) Find the integral by converting to spherical coordinates.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 3e^{(x^2+y^2+z^2)^{3/2}} \, dzdydx$$

Solution: The region of integration is the part of $x^2 + y^2 + z^2 = 1$ in the first octant. The given integral thus equals:

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 3e^{\rho^3} \cdot \rho^2 \sin\phi \, d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} (e-1)\sin\phi \, d\phi d\theta = \int_0^{\pi/2} (e-1) \, d\theta = \frac{(e-1)\pi}{2}$$

2. (C2) For this problem consider the curve $\mathcal{C}$ given by the intersection of the cylinder $y^2+z^2=9$ and the surface $x=yz$, but only the half of this curve with $z \geq 0$.

(a) Parametrize $\mathcal{C}$. Hint: use the fact that $z \geq 0$ to decide the limits on your parameter.

Solution: The part of the circle $y^2+z^2=9$ in the yz-plane with $z \geq 0$ can be described with $y = 3\cos\theta$ and $z = 3\sin\theta$ with $0 \leq \theta \leq \pi$. Using that $x = yz = (3\cos\theta)(3\sin\theta)$ we get parametrization for the curve:

$$\mathbf{r}(\theta) = \langle 9\cos\theta\sin\theta, 3\cos\theta, 3\sin\theta\rangle \text{ with } 0 \leq \theta \leq \pi$$

(b) Compute the vector line integral:

$$\int_{\mathcal{C}} 1\ dx + y\ dy + z\ dz$$

Hint: simplify using some of: $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ or $\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$ or $\sin^2\theta = \frac{1}{2}-\frac{1}{2}\cos 2\theta$.

Solution: We first rewrite our parametrization using the hint as:

$$\mathbf{r}(\theta) = \left\langle \frac{9}{2}\sin 2\theta, 3\cos\theta, 3\sin\theta \right\rangle \text{ with } 0 \leq \theta \leq \pi$$

and then

$$\mathbf{r}'(\theta) = \langle 9\cos 2\theta, -3\sin\theta, 3\cos\theta\rangle$$

So:

$$\int_{\mathcal{C}} 1\ dx + y\ dy + z\ dz = \int_0^{\pi} 9\cos(2\theta) - 9\sin\theta\cos\theta + 9\sin\theta\cos\theta\ d\theta = \int_0^{\pi} 9\cos(2\theta)\ d\theta = \frac{9\sin(\pi) - 9\sin(0)}{2} = 0$$

(c) Set up but **do not compute** a single variable integral that will equal the arclength of $\mathcal{C}$.
Note: the single variable involved should be your parameter.

Solution: It is:

$$\int_{\mathcal{C}} 1\ ds = \int_0^{\pi} \|\mathbf{r}'(\theta)\|\ d\theta = \int_0^{\pi} \sqrt{81\cos^2(2\theta) + 9\sin^2\theta + 9\cos^2\theta}\ d\theta = \int_0^{\pi} 3\sqrt{9\cos^2(2\theta)+1}\ d\theta$$

3. (C3) This problem has multiple parts.

(a) Consider the vector field:

$$\mathbf{F}(x,y,z) = (2xyz)\,\mathbf{i} + \left(x^2 z + e^{z^2}\right)\mathbf{j} + \left(x^2 y + 2yze^{z^2}\right)\mathbf{k}$$

(i) Show that the vector field is conservative by finding a potential function.

Solution: Using guess–and–check I found potential:

$$f(x,y,z) = x^2yz + ye^{z^2}$$

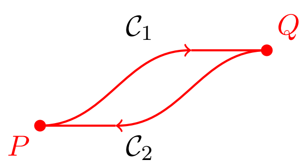
(ii) Consider the path $\mathbf{r}(t) = \langle t+1, t^2-1, e^t\rangle$ with $0 \le t \le 1$ and find:

$$\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{s}$$

Solution: We use the potential function to find:

$$\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{s} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) = f(2,0,e) - f(1,-1,1) = 0 - (-1-e) = 1 + e$$

(b) Consider the oriented curves $\mathcal{C}_1$ and $\mathcal{C}_2$ depicted below.



Suppose that $\mathbf{G}(x,y)$ is a "nice" vector field that is defined and **<u>conservative</u>** everywhere and that:

$$\int_{\mathcal{C}_1} \mathbf{G}\cdot d\mathbf{s} = 4$$

For each part: decide whether there is enough information to determine the value. If there IS enough information, then compute the value. If there IS NOT enough information, then write "undetermined."

(i) $\operatorname{curl}\mathbf{G}(P) =$

(ii) $\displaystyle\int_{\mathcal{C}_2} \mathbf{G}\cdot d\mathbf{s} =$

Solution: (i) Because $\mathbf{G}$ is conservative it follows that $\operatorname{curl}\mathbf{G}(P) = \mathbf{0}$.

(ii) The paths $\mathcal{C}_1$ and $\mathcal{C}_2$ have opposite starts and ends. Because $\mathbf{G}$ is conservative it follows that:

$$\int_{\mathcal{C}_2} \mathbf{G} \cdot d\mathbf{s} = -\int_{\mathcal{C}_1} \mathbf{G} \cdot d\mathbf{s} = -4$$

4. (C4) This problem has multiple parts.

(a) Consider the surface $\mathcal{W}$ (called the Whitney umbrella) parametrized by $\mathbf{X}(u,v) = \langle uv, u, v^2 \rangle$.

(i) Set up **but do not compute** a double integral in u and v that will equal the **surface area** of the portion of $\mathcal{W}$ with $0 \le u \le 2$ and $0 \le v \le 2$. Note: while you do not have to compute the integral, the integrand (expression to integrate) should be computed and simplified.

Solution: We compute:

$$\mathbf{X}_u \times \mathbf{X}_v = \langle v, 1, 0 \rangle \times \langle u, 0, 2v \rangle = \langle 2v, -2v^2, -u \rangle$$

and so:

$$\iint_{\mathbf{X}} 1 \; dS = \int_0^2 \int_0^2 \|\mathbf{X}_u \times \mathbf{X}_v\| \; dudv = \int_0^2 \int_0^2 \sqrt{4v^2 + 4v^4 + u^2} \; dudv$$

(ii) Find an equation of the tangent plane to the Whitney umbrella $\mathcal{W}$ at the point $(2, 1, 4)$.
Note: your final answer should be an equation of a plane!

Solution: First we set up $(2,1,4) = (uv, u, v^2) \implies (u,v) = (1,2)$.
Next: a normal to the surface at $(2,1,4)$ can be found with the help of previous work:

$$\mathbf{X}_u(1,2) \times \mathbf{X}_v(1,2) = \langle 2v, -2v^2, -u \rangle|_{(u=1,v=2)} = \langle 4, -8, -1 \rangle$$

An equation of the plane is thus:

$$4(x-2) - 8(y-1) - 1(z-4) = 0$$

(b) Let $\mathcal{S}$ be the portion of the paraboloid $z = 5 - x^2 - y^2$ with $z \ge 1$ and oriented with **upwards normals**. Find:

$$\iint_{\mathcal{S}} \left\langle x, y, x^2 + y^2 \right\rangle \cdot d\mathbf{S}$$

Note: For a graph $z = f(r)$ it will turn out that $\mathbf{X}_r \times \mathbf{X}_\theta = \langle -f'(r) r \cos\theta, -f'(r) r \sin\theta, r \rangle$. You may use this.

Solution: Note that in cylindrical coordinates, the surface is $z = 5 - r^2$ in which case $z \ge 1$ occurs when $5 - r^2 \ge 1 \implies 4 \ge r^2 \implies 2 \ge r$. We parametrize the surface using:

$$\mathbf{X}(r,\theta) = \langle r\cos\theta, r\sin\theta, 5 - r^2 \rangle \;\text{ with } 0 \le r \le 2 \text{ and } 0 \le \theta \le 2\pi$$

Then:

$$\mathbf{X}_r \times \mathbf{X}_\theta = \langle \cos\theta, \sin\theta, -2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

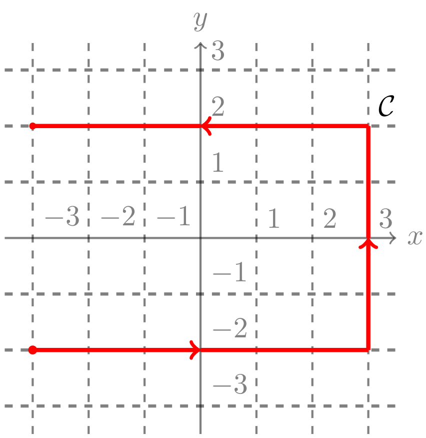
which are upwards as desired. So:

$$\iint_{\mathcal{S}} \left\langle x, y, x^2 + y^2 \right\rangle \cdot d\mathbf{S} = \int_0^2 \int_0^{2\pi} \langle r\cos\theta, r\sin\theta, r^2 \rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle \; d\theta dr = \cdots$$

$$\cdots = \int_0^2 \int_0^{2\pi} 3r^3 \; d\theta dr = 2\pi \cdot \frac{3}{4} \cdot 2^4 = 24\pi$$

5. (C5) This problem has multiple parts.

(a) Let $\mathcal{C}$ be the oriented curve depicted below and find the given integral. Hint: a direct approach is not recommended. Use Green's Theorem by closing off the curve.



$$\int_{\mathcal{C}} \left[4y + \sin\left(e^{e^x}\right)\right] dx + [6x]\, dy$$

Solution: Let $\mathcal{C}'$ be the line segment from $(-3, 2)$ to $(-3, -2)$. Then $\mathcal{C} + \mathcal{C}'$ is counterclockwise closed and so we can compute an integral of the given vector field over this curve using Green's theorem:

$$\int_{\mathcal{C}+\mathcal{C}'} \left[4y + \sin\left(e^{e^x}\right)\right] dx + \left[6x\right] dy = \iint_{\text{enclosed}} (6-4)\ dA = 2 \cdot \text{area(enclosed)} = 48$$

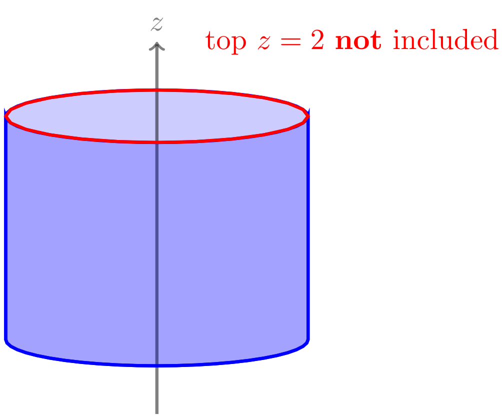
Next we can compute the integral of $-\mathcal{C}'$ using parametrization $\mathbf{r}(y) = \langle -2, y \rangle \implies \mathbf{r}'(y) = \langle 0, 1 \rangle$ with $-2 \le y \le 2$ as:

$$\int_{-\mathcal{C}'} \left[4y + \sin\left(e^{e^x}\right)\right] dx + [6x]\, dy = \int_{-2}^{2} -18\ dy = -72$$

So:

$$\int_{\mathcal{C}} = \int_{\mathcal{C}+\mathcal{C}'} + \int_{-\mathcal{C}'} = 48 - 72 = -24$$

(b) Let $\mathcal{S}$ be the surface consisting of the portion of the cylinder $x^2 + y^2 = 4$ between $-2 \le z \le 2$ along with its bottom at $z = -2$, but **not** its top. Orient $\mathcal{S}$ with **inward** normals. Find the given integral. Hint: Stokes.



$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \ \text{ if } \ \mathbf{F} = -yz\mathbf{i} + xz\mathbf{j} + e^{x^2y^2z^2}\,\mathbf{k}$$

Solution: The boundary ∂S of $\mathcal{S}$ is its top edge : $x^2 + y^2 = 4$ with $z = 2$. To be oriented compatibly with the given orientation, ∂S should be counterclockwise oriented (when viewed from above). We parametrize it as $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 2\rangle \implies \mathbf{r}'(t) = \langle -2\cos t, 2\cos t, 0\rangle$ with $0 \leq t \leq 2\pi$. Then by Stokes's Theorem:

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \langle -4\sin t, 4\cos t, \star\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle \; d\theta = \int_0^{2\pi} 8 \; d\theta = 16\pi$$

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