1. (A1) (4 points) This problem has multiple parts.

(a) If $\|\mathbf{v}\| = 1$ and $\|\mathbf{w}\| = \sqrt{2}$ and $\mathbf{v} \cdot \mathbf{w} = 1$ then find the following.

(i) the angle θ between $\mathbf{v}$ and $\mathbf{w}$

Solution: Since $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$ we have:

$$1 = (1)(\sqrt{2})\cos\theta \implies \frac{1}{\sqrt{2}} = \cos\theta \implies \theta = \frac{\pi}{4}$$

(ii) $\text{proj}_{\mathbf{v}}(\mathbf{w}) \cdot \mathbf{w}$

Solution: We have:

$$\text{proj}_{\mathbf{v}}(\mathbf{w}) \cdot \mathbf{w} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \cdot \mathbf{w} = \left(\frac{1}{1}\right) 1 = 1$$

(iii) $\|(2\mathbf{v} - 3\mathbf{w}) \times (\mathbf{v} + \mathbf{w})\|$

Solution: We have:

$$\|(2\mathbf{v} - 3\mathbf{w}) \times (\mathbf{v} + \mathbf{w})\| = \|2(\mathbf{v} \times \mathbf{w}) - 3(\mathbf{w} \times \mathbf{v})\| = \|5(\mathbf{v} \times \mathbf{w})\| = 5\|\mathbf{v}\|\|\mathbf{w}\| \sin\left(\frac{\pi}{4}\right) = 5(1)(\sqrt{2})\frac{1}{\sqrt{2}} = 5$$

(b) Find the volume of the parallelepiped formed by $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle -2, 1, 4\rangle$ and $\mathbf{c} = \langle 1, 1, 1\rangle$. Note: you do not have to simplify.

Solution: The volume is the absolute value of the scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 0 & 1 \\ -2 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix} = (-3)(1) - 0 + 1(-2 - 1) = -6$$

So the volume is $|-6| = 6$.

2. (A2) (4 points) Consider the lines:

$$\ell_1 : \mathbf{x}(t) = \langle 2+3t, 5-t, 4t \rangle$$

$$\ell_2 : \mathbf{x}(t) = \langle 3+2t, 1-2t, 1+3t \rangle$$

(a) Show that these lines are **not** parallel.

Solution: Respective direction vectors for ℓ_1 and ℓ_2 are $\mathbf{v}_1 = \langle 3, -1, 4 \rangle$ and $\mathbf{v}_2 = \langle 2, -2, 3 \rangle$. These are not scalar multiples of each other, hence the direction vectors are not parallel, hence the lines are not parallel.

(b) Find the (shortest) distance between these two lines. Note: you do not have to simplify.

Solution: We find a common normal to the lines by taking the cross product of their direction vectors:

$$\mathbf{n} = \langle 3, -1, 4 \rangle \times \langle 2, -2, 3 \rangle = \langle 5, -1, -4 \rangle$$

Next we find a point on each line. We have $P = (2, 5, 0)$ on ℓ_1 and $Q = (3, 1, 1)$ on ℓ_2. Next we compute $\mathbf{PQ} = \langle 1, -4, 1 \rangle$. The distance is:

$$|\text{comp}_{\mathbf{n}}(\mathbf{PQ})| = \left| \frac{\mathbf{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|} \right| = \left| \frac{5}{\sqrt{42}} \right| = \frac{5}{\sqrt{42}}$$

(c) Parametrize the plane $\mathcal{P}$ containing all points on the line ℓ_1 and also containing point $A(1, 1, 1)$.
Note: you are being asked to parametrize, **not** for a scalar equation. Your answer should be in terms of two parameters, e.g. s and t. It should **not** be an equation in x, y, and z.

Solution: We have a point $A(1, 1, 1)$ on the line. We need two direction vectors for the plane. One direction vector is the direction vector for ℓ_1: it is $\mathbf{v}_1 = \langle 3, -1, 4 \rangle$. Another direction vector could be the displacement vector from a point on the line ℓ_1 and the point $A(1, 1, 1)$. If we use $P = (2, 5, 0)$ on ℓ_1 then that displacement vector is $\mathbf{PA} = \langle -1, -4, 1 \rangle$. The parametrization is:

$$\mathbf{x}(s, t) = A + s\mathbf{v}_1 + t\mathbf{PA} = \langle 1 + 3s - t, 1 - s - 4t, 1 + 4s + t \rangle$$

3. (A3) (4 points) This problem has multiple parts.

(a) Name the surface $\mathcal{S}$ defined by $x^2 - y^2 + z^2 = 0$. No work required.

paraboloid saddle ellipsoid/sphere 1-sheeted hyperboloid 2-sheeted hyperboloid cone

Solution: It is a cone. You could see this by adding y^2 to both sides to find $x^2 + z^2 = y^2$.

(b) If $\mathbf{r}(0) = \langle 1, 1, 1\rangle$ and $\mathbf{r}'(0) = \langle 1, 2, 3\rangle$ and $f(t) = \mathbf{r}(t) \cdot \mathbf{r}(t)$ then find $f'(0)$.

Solution: We have:

$$f'(0) = \left(\mathbf{r}'(0) \cdot \mathbf{r}(0)\right) + \left(\mathbf{r}(0) \cdot \mathbf{r}'(0)\right) = 2\left(\mathbf{r}'(0) \cdot \mathbf{r}(0)\right) = 2\langle 1, 2, 3\rangle \cdot \langle 1, 1, 1\rangle = 12$$

(c) Let ℓ be the tangent line at point $P(3, 1, 0)$ to the curve $\mathcal{C}$ with parametrization:

$$\mathbf{r}(t) = \langle t^2 + 2, e^{t^2-1}, t + 1\rangle$$

Parametrize this tangent line ℓ. Note: the point P **IS** on the curve.

Solution: First we find the parameter that yields this point. The system:

$$t^2 + 2 = 3$$
$$e^{t^2-1} = 1$$
$$t + 1 = 0$$

yields $t = -1$. Next we get the tangent vector at P:

$$\mathbf{r}'(-1) = \langle 2(-1), 2(-1)e^0, 1\rangle = \langle -2, -2, 1\rangle$$

A parametrization for the tangent line is:

$$P + t\mathbf{r}'(-1) = \langle 3 - 2t, 1 - 2t, 0 + 1t\rangle$$

4. (A4) (4 points) This problem has multiple parts.

(a) Find $h_{xxyx}(x, y)$ if:

$$h(x, y) = (1 + x + 2y)^4$$

Hint: no trick, just calculate.

Solution: We get:

$$h_x = 4(1 + x + 2y)^3 \implies h_{xx} = 12(1 + x + 2y)^2 \implies h_{xxy} = 48(1 + x + 2y) \implies h_{xxyx} = 48$$

(b) Find all (x, y) at which the tangent plane to the graph of $z = g(x, y)$ is **orthogonal** to the plane $x + y + z = 1$, where:

$$g(x, y) = x^2 + xy$$

Hint: Your final answer should turn out to be an equation of a line.

Solution: A normal to the surface g at (x, y) is:

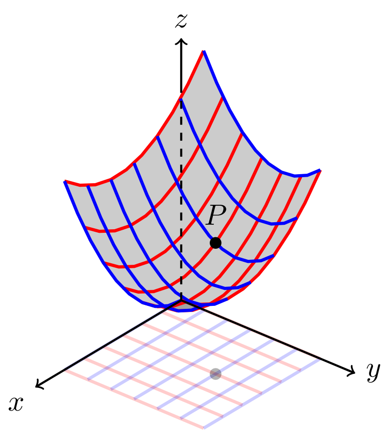
$$\mathbf{n}(x, y) = \langle -g_x, -g_y, 1\rangle = \langle -2x - y, -x, 1\rangle$$

A normal to the plane $x + y + z = 1$ is $\langle 1, 1, 1\rangle$. These normal vectors need to be orthogonal, which means:

$$\langle -2x + y, -x, 1\rangle \cdot \langle 1, 1, 1\rangle = 0 \implies -2x - y - x + 1 = 0 \implies 1 = 3x + y$$

This an equation of a line.

(c) Consider the surface $z = f(x, y)$ sketched below and the point $P = (A, f(A))$. Based on the sketch, decide the signs of the following values. No work required.



$f_x(A)$ is: **positive** **negative**

$f_y(A)$ is: **positive** **negative**

Solution: $f_x(A)$ and $f_y(A)$ are both positive because as x (or y) increases from the point P along the blue x-curve (or red y-curve), the height f is increasing.

Note: the shadows of these curves in the xy-plane are provided. The red curves are y-curves (where x is held constant) and the blue curves are x-curves (where y is held constant)

5. (A5) (4 points) Let $f(x, y, z) = x^2yz$ for all parts of this problem.

(a) Find the directional derivative of f at point $P(1, 2, 3)$ in the unit direction pointing in the same direction as $\mathbf{v} = \langle 2, 1, 2\rangle$.

Solution: First we replace $\mathbf{v}$ with a unit vector:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

Next we compute:

$$\nabla f(1, 2, 3) = \langle 2(1)(2)(3), (1)^2(3), (1)^2(2)\rangle = \langle 12, 3, 2\rangle$$

So:

$$D_{\mathbf{u}} f(1, 2, 3) = \langle 12, 3, 2\rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle = 8 + 1 + \frac{4}{3} = \frac{31}{3}$$

(b) If $\mathbf{x}(t) = \langle 2t - 1, t^2 + 1, t^3 + 2\rangle$ and $g(t) = f(\mathbf{x}(t))$ then find $g'(1)$. Note: f is from the top of the page.

Solution: We find:

$$g'(1) = \nabla f(\mathbf{x}(1)) \cdot \mathbf{x}'(1) = \nabla f(1, 2, 3) \cdot \langle 2, 2, 3\rangle = \langle 12, 3, 2\rangle \cdot \langle 2, 2, 3\rangle = 36$$

(c) Find (if possible) a point of the form $(\pm 1, 2, c)$ on the surface $x^2yz = 2$ where the tangent plane is parallel to the plane $8x + 2y + 4z = 1$. Justifying work is required.

Solution: We plug in $(\pm 1, 2, c)$ into x^2yz to find $(1)(2)c = 2 \implies c = 1$.
Next: the gradient of $f(x, y, z) = x^2yz$ at $(\pm 1, 2, 1)$ is:

$$\nabla f(\pm 1, 2, 1) = \langle \pm 4, 1, 2\rangle$$

The normal to the given plane $8x + 2y + 4z = 1$ is $\langle 8, 2, 4\rangle$. This is parallel to the gradient only when we select $+$ from the $\pm$. So the desired point is $(1, 2, 1)$

6. (B1) (4 points) For this problem let $f(x, y) = 8\ln(x + y) - xy$.

(a) Find all critical points of f and classify them (local min, local max, or saddle). Hint: there is only one. Note: $\ln(\star)$ is not defined if $\star$ is negative. . . this may lead you to eliminate a potential solution.

Solution: We set the partial derivatives equal to zero:

$$f_x(x, y) = \frac{8}{x+y} - y = 0$$
$$f_y(x, y) = \frac{8}{x+y} - x = 0$$

If you subtract the two equations you get $-y + x = 0 \implies y = x$. If we substitute this into the first equation we get:

$$\frac{8}{2x} - x = 0 \implies 4 = x^2 \implies x = \pm 2$$

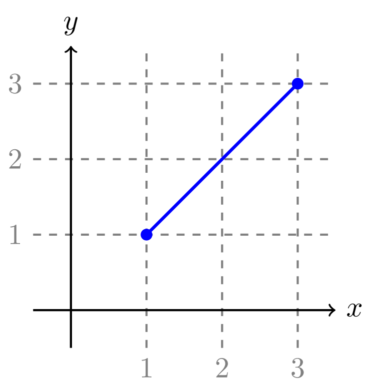
So we obtain candidates $(2, 2)$ and $(-2, -2)$. But $(-2, -2)$ is not in the domain of f and so we ignore it. The Hessian is:

$$Hf(x, y) = \begin{pmatrix} -\frac{8}{(x+y)^2} & -\frac{8}{(x+y)^2} - 1 \\ \frac{8}{(x+y)^2} - 1 & -\frac{8}{(x+y)^2} \end{pmatrix} \implies Hf(2, 2) = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

We find that $\det Hf(2, 2) = \frac{1}{4} - \frac{9}{4} = -2 < 0$ and so $(2, 2)$ is a saddle.

(b) Is the line segment sketched below compact? Yes or no. No work required.

Solution: Yes.



(c) Independent of whether the line segment is compact: you are told that f achieves a maximum value on the line segment. Find the maximum value of f on this line segment. Note: f is from the top of the page.

Note: if you are having trouble comparing values without a calculator: then circle the values you are trying to compare and write something like: "Whatever is biggest!"

Solution: The line is $y = x$ with $1 \le x \le 3$. So along this line we have:

$$f(x,y) = f(x,x) = 8\ln(2x) - x^2 \implies (8\ln(2x) - x^2)' = \frac{8}{x} - 2x \overset{\text{set}}{=} 0 \implies 4 = x^2 \implies x = 2$$

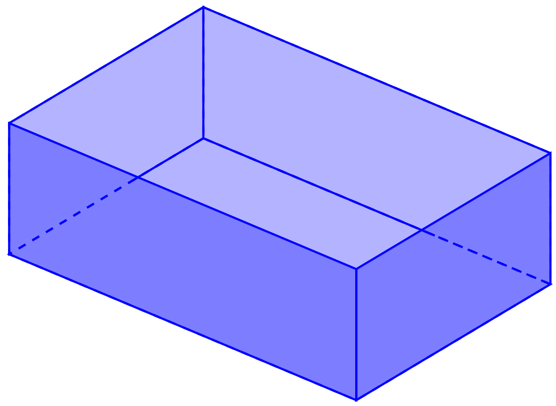
This yields candidate $(2,2)$. The next candidates are the endpoints $(1,1)$ and $(3,3)$. We test:

$$f(2,2) = 8\ln(4) - 4 \text{ and } f(1,1) = 8\ln(2) - 1 \text{ and } f(3,3) = 8\ln(6) - 9$$

The largest of these values is $f(2,2) = 8\ln(4) - 4$, and so this is the absolute maximum value.

7. (B2) (4 points) This problem has two parts.

(a) A box is to be constructed with a volume of 72 cubic inches. The box has four sides and a bottom but no top. The sides of the box cost \$2 per square inch and the bottom of the box costs \$3 per square inch. What are the dimensions of such a box that minimize cost? Use Lagrange multipliers. No other method will be accepted. Note: you may assume that a minimum exists.



Solution: Let x, y, and z be the width, length, and height respectively (all in inches). The goal is to minimize the cost:

$$C = 2(xy) + 3(2yz) + 3(2xz) = 2xy + 6yz + 6xz$$

subject to the constraint $xyz = 72$. We set up Lagrange multipliers:

$$\begin{aligned} 2y + 6z &= \lambda yz \\ 2x + 6z &= \lambda xz \\ 6y + 6x &= \lambda xy \end{aligned}$$

We divide the first two equations to get:

$$\frac{2y+6z}{2x+6z} = \frac{y}{x} \implies 2xy + 6xz = 2xy + 6yz \implies 6xz = 6yz \implies x = y$$

We divide the second two equations to get:

$$\frac{2x+6z}{6y+6x} = \frac{z}{y} \implies 2xy + 6yz = 6yz + 6xz \implies 2xy = 6xz \implies y = 3z$$

Altogether we have $x = y = 3z$. We substitute into the constraint $xyz = 81$ to get:

$$(3z)(3z)z = 72 \implies z^3 = 8 \implies z = 2$$

So the dimensions of the box that minimize cost are $x = 6$ and $y = 6$ and $z = 2$.

(b) The point P is on the level curve $g = 1$ and:

- $\nabla f(P) = \langle 2, 2 \rangle$
- $\nabla g(P) = \langle 1, 1 \rangle$

According to Lagrange multipliers: could P possibly be an extremizer of f subject to the constraint $g = 1$? Briefly explain. Note: an answer without correct explanation receives no credit.

Solution: Yes it could be, because the equation:

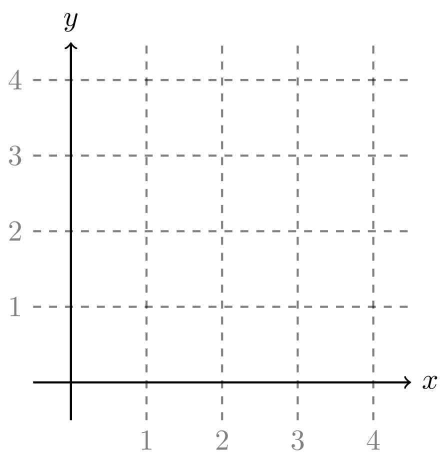
$$\nabla f(P) = \lambda \nabla g(P)$$

has a solution: namely $\lambda = 2$.

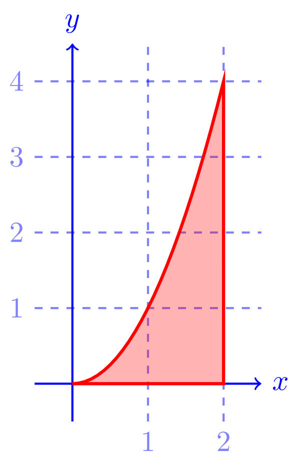
8. (B3) (4 points) This problem has multiple parts.

(a) Find the integral. Hint: the given order of integration is no good.

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{x^3+1}\, dxdy$$



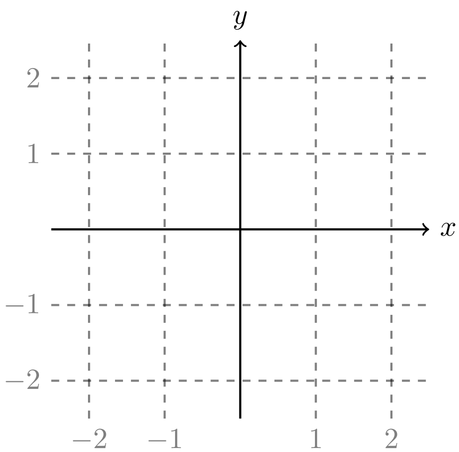
Solution: The region of integration is:



where the curved edge is $x = \sqrt{y} \implies y = x^2$. In the other order of integration our integral is:

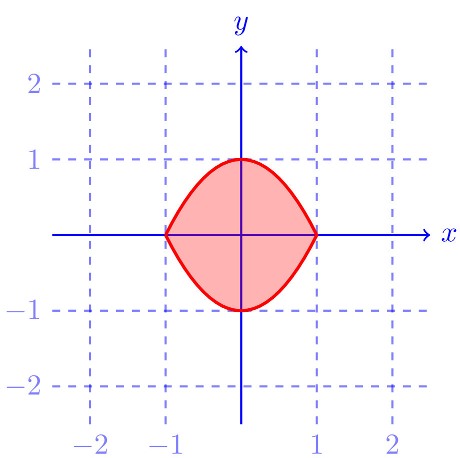
$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{x^3+1}\, dxdy = \int_0^2 \int_0^{x^2} \frac{1}{x^3+1}\, dydx = \int_0^2 \frac{x^2}{x^3+1}\, dx = \frac{1}{3}\left[\ln(x^3+1)\right]_{x=0}^2 = \frac{1}{3}\ln 9$$

(b) Set up **but do not compute** a double integral in order $dydx$ or $dxdy$ (your choice) that equals the volume of the region bounded above by the plane $-x - y + z = 4$ and below by the xy-plane ($z = 0$) and on the sides by $y = x^2 - 1$ and $y = 1 - x^2$.



Solution: We can solve for z in the equation of the top plane: $z = 4 + x + y$. Since the desired volume is between this plane and the xy-plane, we will use $4 + x + y$ as the integrand in our double–integral.

Next we describe the region of integration. Here it is sketched below:

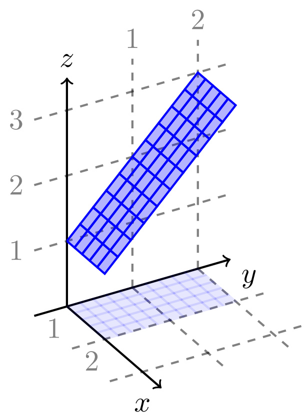


where the curved edge on top is $y = 1 - x^2$ and the one on bottom is $y = x^2 - 1$. So our integral in order $dydx$ is:

$$\int_{-1}^{1} \int_{x^2-1}^{1-x^2} 4 + x + y \, dydx$$

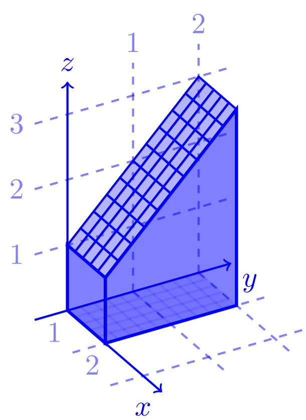
9. (B4) For this problem let E be the combined region of integration of:

$$\int_0^1 \int_1^3 \int_{z-1}^2 f(x,y,z)\ dydzdx + \int_0^1 \int_0^1 \int_0^2 f(x,y,z)\ dydzdx.$$



(a) Rewrite the sum of iterated integrals as a **single** integral in the order $dzdydx$ and **sketch** the region of integration, indicating in the sketch which part comes from the first integral and which part comes from the second.

Solution: Here is the region of integration:



where the top is the plane $y = z - 1 \implies z = 1 + y$. In the order $dzdydx$ the integral is:

$$\int_0^1 \int_0^2 \int_0^{1+y} f(x,y,z)\ dzdydx$$

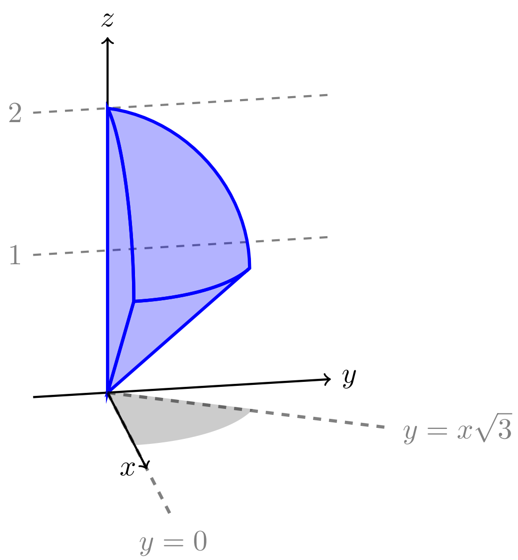
(b) Let $\delta(x,y,z) = 6xz$ be the density of mass at point (x,y,z) in the region E. Find the mass of E.

Solution: We compute:

$$\int_0^1 \int_0^2 \int_0^{1+y} 6xz\ dzdydx = \int_0^1 \int_0^2 3x(1+y)^2\ dydx = \int_0^1 26x\ dx = 13$$

10. (B5) (4 points) This problem has multiple parts.

(a) Let E be the 3D region with $0 \leq y \leq x\sqrt{3}$ and bounded below by the cone $z = \sqrt{x^2+y^2}$ and bounded above by the top half of the shifted sphere $(z-1)^2 + x^2 + y^2 = 1$. Find the bounds if you were to evaluate an integral over E in the order $dzdrd\theta$. Note: cylindrical coordinates only!



$\leq z \leq$

$\leq r \leq$

$\leq \theta \leq$

Solution: The cone is $z = \sqrt{x^2+y^2} = r$ and we can solve for z in the sphere equation to find the top–half of the sphere as $z = 1 + \sqrt{1-x^2-y^2} = 1 + \sqrt{1-r^2}$ and so we get bounds on z as:

$$r \leq z \leq 1 + \sqrt{1-r^2}$$

The surfaces intersect when:

$$r = 1 + \sqrt{1-r^2} \implies r - 1 = \sqrt{1-r^2} \implies r = 1$$

So the bounds on r are:

$$0 \leq r \leq 1$$

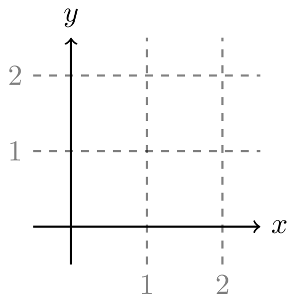
Lastly we are given:

$$0 \leq y \leq x\sqrt{3} \implies 0 \leq r\sin\theta \leq r\cos\theta\sqrt{3} \implies 0 \leq \tan\theta \leq \sqrt{3} \implies 0 \leq \theta \leq \frac{\pi}{3}$$

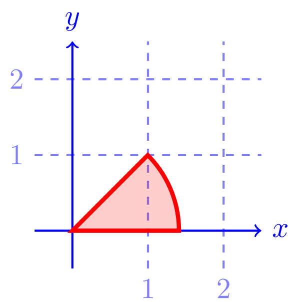
(b) Use polar coordinates to evaluate the following integral.

$$\int_0^1 \int_y^{\sqrt{2-y^2}} \frac{y}{\sqrt{x^2+y^2}}\, dxdy$$

Hint: sketch the region. The polar bounds should turn out to be constants.



Solution: Here is a sketch of the region of integration:



The circle equation is found from the bound:

$$x = \sqrt{2-y^2} \implies x^2+y^2 = 2 \implies r = \sqrt{2}$$

The bound on θ is $0 \le \theta \le \frac{\pi}{4}$. The integral in polar is thus:

$$\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \frac{r\sin\theta}{\not{r}}\, \not{r} dr d\theta = \int_0^{\frac{\pi}{4}} \sin\theta\ d\theta = 1 - \frac{1}{\sqrt{2}}$$

11. (C1) (4 points) This question has multiple parts.

(a) Let E be the region above the paraboloid $z = r^2$ and below the cone $z = \sqrt{3}r$ and with $x, y \geq 0$. Find the bounds in order $d\rho d\phi d\theta$ if E is the region of integration.

$$\leq \rho \leq$$
$$\leq \phi \leq$$
$$\leq \theta \leq$$

Note: r is the cylindrical coordinate. Your answer should have spherical coordinates only.

Note: do not forget the requirement $x, y \geq 0$.

Solution: We convert the cone to spherical to get:

$$z = \sqrt{3}r \implies \rho\cos\phi = \sqrt{3}\rho\sin\phi \implies \cot\phi = \sqrt{3} \implies \phi = \frac{\pi}{6}$$

The region is below the cone but above the xy-plane and so:

$$\frac{\pi}{6} \leq \phi \leq \frac{\pi}{2}$$

We convert the paraboloid to spherical to get:

$$z = r^2 \implies \implies \rho\cos\phi = \rho^2\sin^2\phi \implies \cos\phi = \rho\sin^2\phi \implies \rho = \frac{\cos\phi}{\sin^2\phi}$$

This represents the upper limit of ρ in our region, and therefore:

$$0 \leq \rho \leq \frac{\cos\phi}{\sin^2\phi}$$

Lastly because we are in the first octant we have:

$$0 \leq \theta \leq \frac{\pi}{2}$$

(b) Use integration with spherical coordinates to calculate the volume of the region between the spheres $x^2+y^2+z^2 = 4$ and $x^2 + y^2 + z^2 = 9$ and inside the cone $z = \sqrt{x^2 + y^2}$. Note: you do not need to simplify.

Solution: The spheres given are $\rho = 2$ and $\rho = 3$. So $2 \leq \rho \leq 3$.

The cone is $z = r \implies \rho\cos\phi = \rho\sin\phi \implies \phi = \frac{\pi}{4}$. So in our region $0 \leq \phi \leq \frac{\pi}{4}$.

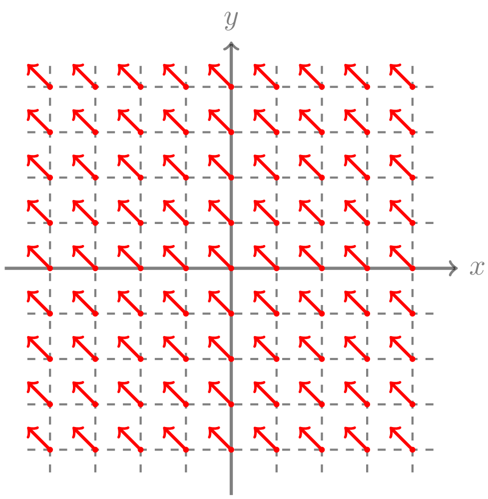
There is no restriction on θ and therefore $0 \leq \theta \leq 2\pi$.

We compute:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_2^3 \rho^2\sin\phi\ d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{19}{3}\sin\phi\ d\phi d\theta = \int_0^{2\pi} \frac{19}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\ d\theta = \frac{38\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

12. (C2) This problem has multiple parts.

(a) Let $\mathbf{F}(x, y)$ be the vector field sketched below. Do the following on the sketch.

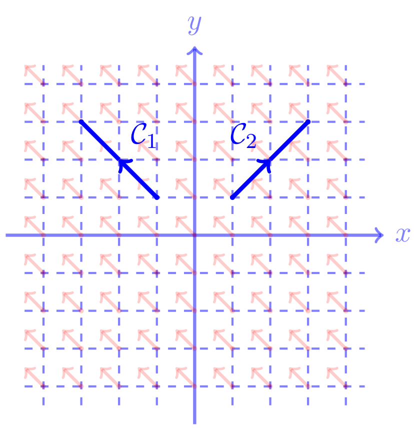


(i) Sketch an oriented curve $\mathcal{C}_1$ so $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} > 0$.

(ii) Sketch an oriented curve $\mathcal{C}_2$ so $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = 0$.

Note: make sure your sketches are labeled and show the directions (orientations) of your curves. Your curves cannot have zero length.

Solution: $\mathcal{C}_1$ needs to point more in the direction of arrows than against them. $\mathcal{C}_2$ could point orthogonally to the direction of the arrows.



(b) Let $\mathcal{C}$ be the straight line segment starting at $(1, 0)$ and ending at $(4, 4)$.

(i) Let $\mathbf{G}(x, y) = \langle x, y \rangle$ and find $\int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s}$.

Solution: We parametrize $\mathcal{C}$ using:

$$\mathbf{r}(t) = \langle 1, 0 \rangle + t\langle 4 - 1, 4 - 0 \rangle = \langle 1 + 3t, 4t \rangle \text{ with } 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = \langle 3, 4 \rangle$$

We find:

$$\int_{\mathcal{C}} \langle x, y \rangle \cdot d\mathbf{s} = \int_0^1 \langle 1 + 3t, 4t \rangle \cdot \langle 3, 4 \rangle \; dt = \int_0^1 3 + 25t \; dt = 3 + \frac{25}{2} = \frac{31}{2}$$

(ii) Find the area of the vertical fence whose base is the curve $\mathcal{C}$ and whose top lies on the surface $z = 32(x-1)e^{y^2}$.

Note: you do not need to simplify.

Solution: We calculate:

$$\|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2} = 5$$

The integral we want to compute is:

$$\int_{\mathcal{C}} 32(x-1)e^{y^2}\ ds = \int_0^1 32(3t)e^{16t^2}\ 5dt = 15\left[e^{16t^2}\right]_{t=0}^{1} = 15[e^{16} - 1]$$

13. (C3) This problem has multiple parts.

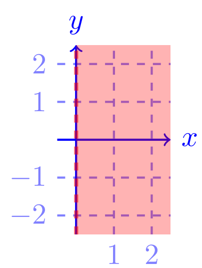
(a) Let a be a constant and consider the vector field:

$$\mathbf{F}(x,y) = \left(\frac{e^{2y}}{x} + \sin(x^2)\right)\mathbf{i} + \left(e^{y^2} + a(\ln x)e^{2y}\right)\mathbf{j}$$

which has domain consisting of all (x,y) with $x > 0$.

(i) Sketch the domain of $\mathbf{F}(x,y)$ in the xy-plane and decide whether it is simply–connected. Provide a brief explanation.

Solution: The region in question is:



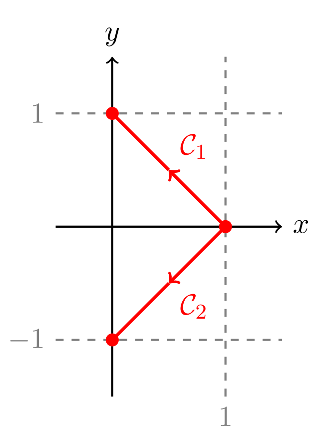
which is simply–connected roughly because it has no holes.

(ii) Find the value of a that guarantees $\mathbf{F}(x,y)$ is conservative and explain why your choice works.

Solution: If the curl of $\mathbf{F}$ is zero, then since its domain is simply–connected, $\mathbf{F}$ will be conservative. We calculate:

$$\operatorname{curl}\mathbf{F} = \left(\partial_x\left[e^{y^2} + a(\ln x)e^{2y}\right] - \partial_y\left[\frac{e^{2y}}{x} + \sin(x^2)\right]\right)\mathbf{k} = \left(\frac{ae^{2y}}{x} - \frac{2e^{2y}}{x}\right)\mathbf{k} \overset{\text{set}}{=} \mathbf{0} \implies a = 2$$

(b) Consider the oriented line segments $\mathcal{C}_1$ and $\mathcal{C}_2$ below.



Let $\mathbf{G}(x,y)$ be a **conservative** vector field so that:

$$\int_{\mathcal{C}_1} \mathbf{G}\cdot d\mathbf{s} = 2 \text{ and } \int_{\mathcal{C}_2} \mathbf{G}\cdot d\mathbf{s} = 3$$

Using only this information and nothing else, find:

$$\int_{\mathbf{x}} \mathbf{G}\cdot d\mathbf{s} =$$

where $\mathbf{x}$ is the path $\mathbf{x}(t) = \langle t^2 - 1, t\rangle$ for $-1 \le t \le 1$

Hint: can you construct another path with the same start and end as $\mathbf{x}(t)$?

Solution: The start of our path is $\mathbf{x}(-1) = \langle 0, -1\rangle$ and the end of our path is $\mathbf{x}(1) = \langle 0, 1\rangle$. The path given by $-\mathcal{C}_2$ followed by $\mathcal{C}_1$ has this same start and end. So, since $\mathbf{G}$ is conservative and has path–independent line integrals, we find:

$$\int_{\mathbf{x}} \mathbf{G} \cdot d\mathbf{s} = -\int_{\mathcal{C}_2} \mathbf{G} \cdot d\mathbf{s} + \int_{\mathcal{C}_1} \mathbf{G} \cdot d\mathbf{s} = -3 + 2 = -1$$

14. (C4) This problem has multiple parts.

(a) Let $\mathcal{S}$ be the portion of the cone $z = 2\sqrt{x^2+y^2}$ with $2 \leq z \leq 4$. Find the surface area of $\mathcal{S}$ using a surface integral. Hint: cylindrical parameters.

Solution:

We parametrize $\mathcal{S}$ using r and θ to find:

$$\mathbf{X}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{4r^2}\rangle = \langle r\cos\theta, r\sin\theta, 2r\rangle$$

with bounds $2 \leq z \leq 4$ that we write in terms of our parameters r as $2 \leq 2r \leq 4 \implies 1 \leq r \leq 2$. Because θ is unrestricted its bounds are $0 \leq \theta \leq 2\pi$.

Next using the hint (our cone is $z = 2r$) we have that:

$$\|\mathbf{X}_r \times \mathbf{X}_\theta\| = \|\langle -2r\cos\theta, -2r\sin\theta, r\rangle\| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = \sqrt{5}r$$

The surface area is:

$$\iint_{\mathcal{S}} 1\ dS = \int_0^{2\pi}\int_1^2 \sqrt{5}r\ drd\theta = 2\pi\cdot\sqrt{5}\cdot\frac{3}{2} = 3\pi\sqrt{5}$$

(b) Let $\mathcal{S}$ be the portion of the graph $z = ye^x$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ oriented with **downward normals**. If:

$$\mathbf{F}(x,y,z) = 1\mathbf{i} + 1\mathbf{j} + z\mathbf{k} \text{ then find: } \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$$

Solution: We parametrize with parameters x and y:

$$\mathbf{X}(x,y) = \langle x, y, ye^x\rangle \text{ with } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

Then the induced normal is:

$$\mathbf{X}_x \times \mathbf{X}_y = \langle -ye^x, -e^x, 1\rangle$$

This is upwards, so we will introduce a negative when computing our integral, because it required downward normals. We find:

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = -\int_0^1\int_0^1 \langle 1, 1, ye^x\rangle\cdot\langle -ye^x, -e^x, 1\rangle\ dydx = -\int_0^1\int_0^1 -e^x\ dydx = e - 1$$

15. (C5) This problem has multiple parts.

(a) Set up **but do not compute** a single variable integral that equals the **area** of the region enclosed by the path:

$$\mathbf{x}(t) = \langle \cos t - \cos^2 t, \sin t - \sin t \cos t \rangle \text{ with } 0 \leq t \leq 2\pi$$

Hint: involve Green's Theorem by converting to an appropriate vector line integral.
Note: your final answer should look like $\int_?^? \; ? \; dt$ where each ? has been found **in terms of t and constants!**

Solution: Let $\langle P, Q \rangle = \langle 0, x \rangle$ (though there are other correct choices for P and Q). By Green's Theorem:

$$\int_{\mathbf{x}} \langle 0, x \rangle \cdot d\mathbf{s} = \iint_{\text{enclosed}} \partial_x[x] - \partial_y[0] \; dA = \iint_{\text{enclosed}} 1 \; dA = \text{area(enclosed)}$$

Alright: so let's set up the vector line integral all the way on the left. We first find:

$$\mathbf{x}'(t) = \langle \star, \cos t - \cos^2 t + \sin^2 t \rangle$$

and so:

$$\int_{\mathbf{x}} \langle 0, x \rangle \cdot d\mathbf{s} = \int_0^{2\pi} \langle 0, \cos t - \cos^2 t \rangle \cdot \langle \star, \cos t - \cos^2 t + \sin^2 t \rangle \; dt = \int_0^{2\pi} (\cos t - \cos^2 t)(\cos t - \cos^2 t + \sin^2 t) \; dt$$

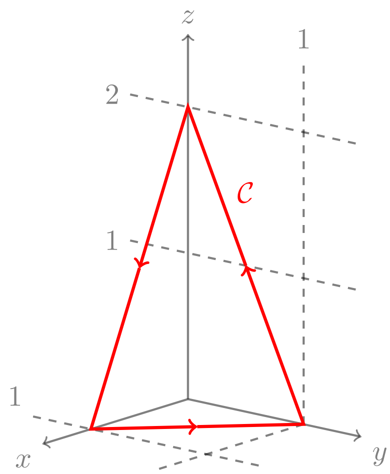
(b) Let $\mathcal{C}$ be the oriented curve of straight line segments:

$$(1,0,0) \to (0,1,0) \to (0,0,2) \to (1,0,0)$$

and use Stokes's Thereom to find:

$$\int_{\mathcal{C}} \Big(x\sin(e^x) - z\Big)\ dx - 2y\ dy + (y-z)\ dz$$

Hint: this triangle is on the plane $z = 2 - 2x - 2y$.



Solution: Let $\mathcal{S}$ be the triangle with these segments as its edges. The curve $\mathcal{C}$ is counterclockwise as viewed from above, and therefore the normals that will be compatible with Stokes's Theorem are upwards.

Next we parametrize $\mathcal{S}$ using parameters x and y to get:

$$\mathbf{X}(x,y) = \langle x, y, 2-2x-2y\rangle \text{ with } (x,y) \text{ in the triangle with vertices } (0,0),\ (1,0),\ (0,1)$$

then:

$$\mathbf{X}_x \times \mathbf{X}_y = \langle 2,2,1\rangle$$

this is upwards, so compatible with Stokes.

Let $\mathbf{F} = \langle x\sin(e^x) - z, -2y, y-z\rangle$ be the vector field being integrated. Then:

$$\operatorname{curl}\mathbf{F} = \langle 1,-1,0\rangle$$

We are ready to compute the integral:

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{xy\text{-shadow}} \langle 1,-1,0\rangle\cdot\langle 2,2,1\rangle\ dA = \iint_{xy\text{-shadow}} 0\ dA = 0$$

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