Math 226 Retake ≤B

Fa24

Tue Nov 12

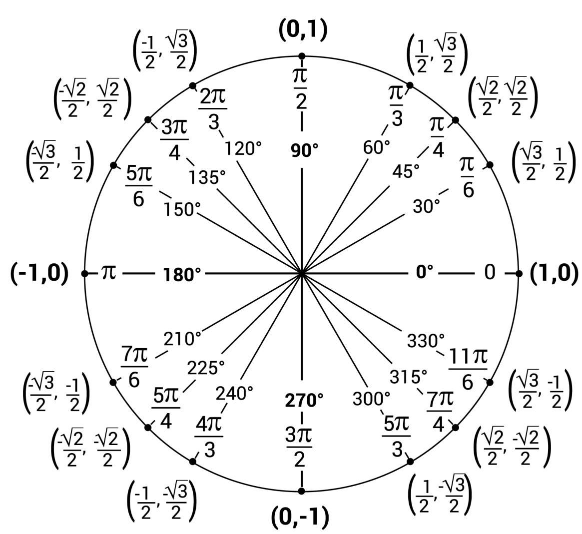
firstname lastname: ______________________________

uscid: ______________________________

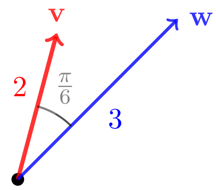
Instructions

- Before the examination starts: enter your old scores into the table below.
- This examination consists of 14 pages not including this cover page. Verify that your copy of this examination contains all 14 pages. If your examination is missing any pages then obtain a new copy immediately.
- You have 50 minutes to complete this examination.
- Do not use books, calculators, computers, tablets, or phones. You may use both sides of a single 3 in by 5 in index card but no other notes.
- Write legibly in the boxed area only. Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.
- If you run out of space: there are two pages at the end where you can continue your work.

Topic	A1	A2	A3	A4	A5	B1	B2	B3	B4	B5
Old Score										
Max	4	4	4	4	4	4	4	4	4	4



1. (A1) (4 points) For all parts of this problem consider $\mathbf{v}$ and $\mathbf{w}$ sketched below.



Find the following without referring to the components (entries) of the vectors. You do not have to simplify your answer as long as it only involves numbers (and maybe trig functions).

(a) $(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) =$

Solution:

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta + \|\mathbf{w}\|^2 = 4 + 12\cos\left(\frac{\pi}{6}\right) + 9 = 13 + 6\sqrt{3}$$

(b) $\|\mathbf{v}\times\mathbf{w}\| =$

Solution:

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta = 6\sin\left(\frac{\pi}{6}\right) = 3$$

(c) the scalar component of of $\mathbf{w}$ along $\mathbf{v}$, i.e. $\operatorname{comp}_{\mathbf{v}}(\mathbf{w}) =$

Solution:

$$\operatorname{comp}_{\mathbf{v}}(\mathbf{w}) = \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta}{\|\mathbf{v}\|} = \|\mathbf{w}\|\cos\theta = 3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$$

(d) Decide whether $\mathbf{v}\times\mathbf{w}$ points **into** the page or **out of** the page. No work required.

Solution: Into the page.

2. (A2) (4 points) Let $\mathcal{P}_1$ be the plane thru $A(1,1,1)$ and orthogonal to $\mathbf{n}_1 = \langle 3,2,1\rangle$ and let $\mathcal{P}_2$ be the plane with equation $x + 2y + 3z = 6$.

(a) Find a parametrization for the line ℓ of intersection of the two planes. Hint: the point $A(1,1,1)$ is also on $\mathcal{P}_2$ and so would be on this line of intersection. Now you need a direction vector $\mathbf{v}$: what should the relation of this direction vector be to the normal vectors to our planes?

Solution: Because ℓ is on both planes, its direction vector is orthogonal to the normal vectors of both planes. We can thus find a direction vector $\mathbf{v}$ by taking a cross product of the normal vectors of the planes:

$$\mathbf{v} = \langle 3,2,1\rangle \times \langle 1,2,3\rangle = \langle 4,-8,4\rangle$$

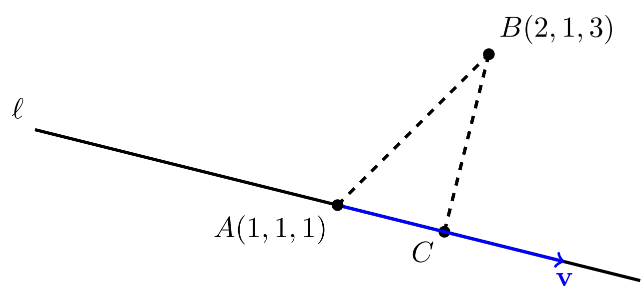
which we can rescale for fun to:

$$\langle 1,-2,1\rangle$$

Since the hint tells us $A(1,1,1)$ is on the line, a parametrization for the line ℓ is:

$$A + t\mathbf{v} = \langle 1+t, 1-2t, 1+t\rangle$$

(b) Find the closest point C on the line ℓ (from the previous part) to the point $B(2,1,3)$. Hint: I've included a suggestive, but not to scale, sketch using the direction vector $\mathbf{v}$ you found for ℓ in part (a). Note that the dashed line segment from B to C is **orthogonal** to ℓ.



Note: your final answer should be the coordinates of the point C. You do not need to simplify.

Solution: We can find $\mathbf{AC}$ using the orthogonal projection of $\mathbf{AB} = \langle 1,0,2\rangle$ onto $\mathbf{v} = \langle 1,-2,1\rangle$. We get:

$$\mathbf{AC} = \text{proj}_{\mathbf{v}}(\mathbf{AB}) = \left(\frac{\mathbf{v}\cdot\mathbf{AB}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{3}{6}\langle 1,-2,1\rangle = \langle 0.5,-1,0.5\rangle$$

So:

$$C = A + \mathbf{AC} = (1.5, 0, 1.5)$$

3. (A3) (4 points) This problem has multiple parts.

(a) Name the surface $\mathcal{S}$ defined by $4(x-1)^2+(y-2)^2+4(z-3)^2=4$. No work required.

paraboloid saddle ellipsoid/sphere 1-sheeted hyperboloid 2-sheeted hyperboloid cone

Solution: ellipsoid

(b) Let ℓ be the tangent line at point $P(8,-5,10)$ to the curve $\mathcal{C}$ with parametrization:

$$\mathbf{r}(t)=\langle t^2-1, 2t+1, t^2+1\rangle$$

Parametrize this tangent line ℓ. Hint: the point P **IS** on the curve. Maybe first find the value of the parameter yielding this point.

Solution: We first solve for the t that yields point P. Solving:

$$\begin{aligned} t^2-1&=8\\ 2t+1&=-5\\ t^2+1&=10 \end{aligned}$$

yields $t=-3$. Next we get the tangent vector at P:

$$\mathbf{r}'(-3)=\langle 2(-3),2,2(-3)\rangle=\langle -6,2,-6\rangle$$

A parametrization for the tangent line is:

$$P+t\mathbf{r}'(-3)=\langle 8-6t,-5+2t,10-6t\rangle$$

(c) Suppose that suppose that $\mathbf{r}(t)=\mathbf{r}''(t)$. Show that:

$$\frac{d}{dt}\Big[\mathbf{r}(t)\times\mathbf{r}'(t)\Big]=\mathbf{0}$$

Hint: product rule. Don't forget to use the given information: $\mathbf{r}(t)=\mathbf{r}''(t)$.

Solution: We compute:

$$\frac{d}{dt}\Big[\mathbf{r}(t)\times\mathbf{r}'(t)\Big]=\big(\mathbf{r}'(t)\times\mathbf{r}'(t)\big)+\big(\mathbf{r}(t)\times\mathbf{r}''(t)\big)=\mathbf{0}+\mathbf{r}''(t)\times\mathbf{r}''(t)=\mathbf{0}+\mathbf{0}=\mathbf{0}$$

4. (A4) (4 points) For this problem consider the function $f(x, y) = \sqrt{9 + 4x + 6y}$.

(a) Parametrize the **normal** line to the graph of $z = f(x, y)$ at the point $P(1, 2, 5)$. Hint: you need to find a normal vector to this surface. To make it easier to plug in: the simplified derivative of $\sqrt{u}$ is $\frac{1}{2\sqrt{u}}$.

Solution: At $P(1, 2, 5)$ we have $x = 1$ and $y = 2$. So we look at:

$$f_x(1, 2) = \frac{4}{2\sqrt{9 + 4(1) + 6(2)}} = \frac{2}{5} \quad \text{and} \quad f_y(1, 2) = \frac{6}{2\sqrt{9 + 4(1) + 6(2)}} = \frac{3}{5}$$

a normal vector to the tangent plane at P is:

$$\mathbf{n} = \langle -f_x(1, 2), -f_y(1, 2), 1\rangle = \left\langle -\frac{2}{5}, -\frac{3}{5}, 1 \right\rangle$$

Hence a parametrization for the normal line is:

$$P + t\mathbf{n} = \left\langle 1 - \frac{2t}{5}, 2 - \frac{3t}{5}, 5 + t \right\rangle$$

(b) Use the idea of a linear approximation to estimate:

$$f(1.2, 1.8) \approx$$

You do not need to simplify. Hint: use your work from part (a).

Solution: The equation of the tangent plane at $x = 1$ and $y = 2$ is (using our work from part (a)):

$$z = 5 + \frac{2}{5}(x - 1) + \frac{3}{5}(y - 2)$$

We substiute $x = 1.2$ and $y = 1.8$ to get:

$$z = 5 + \frac{2}{5}(0.2) + \frac{3}{5}(-0.2) \approx f(1.2, 1.8)$$

(c) You are told that:

$$f_{xyy}(x, y) = 54(7 + 4x + 6y)^{-5/2}$$

Find, and don't worry about simplifying: $f_{yyxx}(x, y) =$

Solution: To obtain f_{yyxx} from f_{xyy} we just need to take one more x derivative. We find:

$$\frac{\partial}{\partial y}\left[54(7 + 4x + 6y)^{-5/2}\right] = -\frac{5}{2} \cdot 4 \cdot 54(7 + 4x + 6y)^{-7/2}$$

5. (A5) (4 points) This problem has multiple parts.

(a) If $x = 2s + t$ and $y = s + 2t$ and $z = f(x, y)$ then show that:

$$\frac{\partial z}{\partial s} + \frac{\partial z}{\partial t} = 3\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)$$

Hint: compute the left side using the chain rule and then compare against the right side.

Solution: Using the chain rule we find:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}\cdot 2 + \frac{\partial z}{\partial y}\cdot 1$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}\cdot 1 + \frac{\partial z}{\partial y}\cdot 2$$

When we add these we find:

$$\frac{\partial z}{\partial s} + \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot 2 + \frac{\partial z}{\partial y}\cdot 1 + \frac{\partial z}{\partial x}\cdot 1 + \frac{\partial z}{\partial y}\cdot 2 = 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 3\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)$$

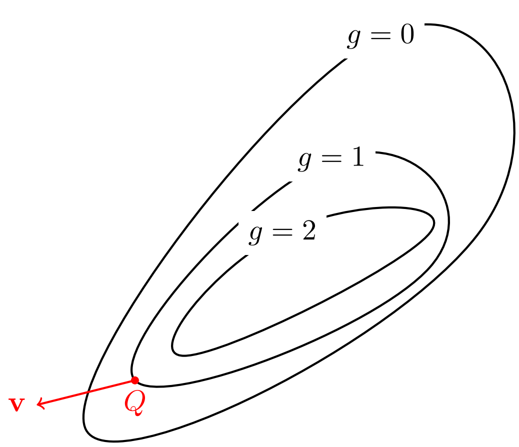
(b) Find an equation of the tangent plane to the surface $x^2 - yz = 1$ at the point $P(2, 1, 3)$.

Solution: Let $f(x, y, z) = x^2 - yz$. A normal vector to the surface at $P(2, 1, 3)$ is:

$$\nabla f(2, 1, 3) = \langle 2(2), -(3), -(1)\rangle = \langle 4, -3, -1\rangle$$

The tangent plane has equation $4(x - 2) - 3(y - 1) - 1(z - 3) = 0$.

(c) Consider the contour diagram for a function g given below. No work required.



(i) **Sketch** the gradient $\nabla g(Q)$ on the diagram. Its length is not important, but its direction is.

Solution: When rooted at Q, points orthogonal to level curve $g = 1$, but towards $g = 2$ level curve.

(ii) Consider the vector $\mathbf{v}$ sketched in the diagram. Decide whether the directional derivative $D_{\mathbf{v}}g(Q)$ is:

positive **negative** **zero**

Solution: negative because the angle between $\mathbf{v}$ and $\nabla g(Q)$ is greater than 90° or alternatively because $\mathbf{v}$ is pointing towards a smaller level curve.

6. (B1) (4 points) For this problem let $f(x,y) = 2y^3 + 3y^2 + 12xy + 3x^2 - 6x$.

(a) **Find all** critical points of f and **classify the one with largest y-coordinate** (as local min, local max, or saddle). Note: I only ask you to **classify one** to save you time, but you still need to **find all** critical points.

Hint: There are two critical points. Some of the expressions that you are trying to assess the sign of may be easier to deal with if you factor out 6.

Solution: We set the partial derivatives equal to zero:

$$f_x(x,y) = 12y + 6x - 6 = 0 \implies x = 1 - 2y$$
$$f_y(x,y) = 6y^2 + 6y + 12x = 0 \implies y^2 + y + 2x = 0$$

We substitute $x = 1 - 2y$ into the second equation to get:

$$y^2 + y + 2 - 4y = 0 \implies y^2 - 3y + 2 = 0 \implies (y-1)(y-2) = 0$$

So we find $y = 1 \implies x = -1$ or $y = 2 \implies x = -3$.

Our critical points are $(-1,1)$ and $(-3,2)$.

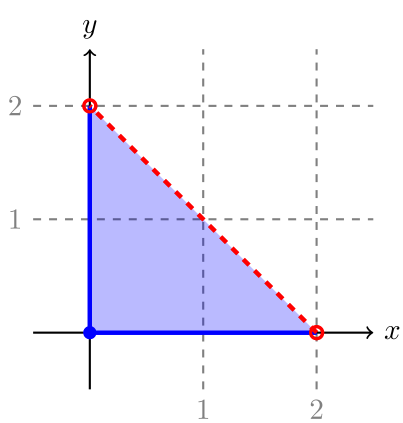
The Hessian is:

$$Hf(x,y) = \begin{pmatrix} 6 & 12 \\ 12 & 12y+6 \end{pmatrix} \implies Hf(-1,1) = \begin{pmatrix} 6 & 12 \\ 12 & 18 \end{pmatrix} \text{ and } Hf(-3,2) = \begin{pmatrix} 6 & 12 \\ 12 & 30 \end{pmatrix}$$

We find that $\det Hf(-1,1) = 6 \cdot 18 - 12 \cdot 12 = 6(18-24) = -36 < 0$ and so $(-1,1)$ is a saddle.

We find that $\det Hf(-3,2) = 6 \cdot 30 - 12 \cdot 12 = 6(30-24) = 36 > 0$ and that $\operatorname{tr} Hf(-3,2) = 36 > 0$. So $(-3,2)$ is a local minimum.

(b) Find the absolute minimum value of f on the solid triangle (depicted below) with the help of this fact: the absolute minimum does **not** occur on the diagonal edge of the triangle (dashed red). Note: You may assume a minimum exists.



Solution: Neither of the critical points we found in part (a) is feasible, and so both can be ignored for this part.

One candidate is the corner $(0, 0)$.

Next we parametrize the bottom edge: $(x, 0)$ with $0 \leq x \leq 2$. We find:

$$f(x, 0) = 3x^2 - 6x \implies (3x^2 - 6x)' = 6x - 6 \overset{\text{set}}{=} 0 \implies x = 1$$

So we have candidate $(1, 0)$.

Next we parametrize the left edge: $(0, y)$ with $0 \leq y \leq 2$. We find:

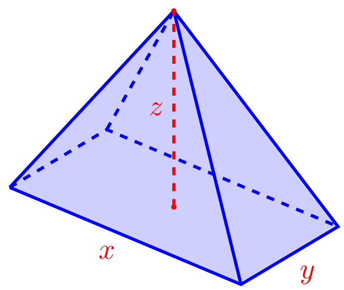
$$f(0, y) = 2y^3 + 3y^2 \implies (2y^3 + 3y^2)' = 6y^2 + 6y = 6y(y + 1) \overset{\text{set}}{=} 0 \implies y = 0 \text{ or } y = -1$$

We ignore $y = -1$ because it is not feasible. If $y = 0$ we get $(0, 0)$, the corner.

So our only candidates are $(0, 0)$ and $(1, 0)$. We find: $f(0, 0) = 0$ and $f(1, 0) = 3 - 6 = -3$. So the minimum is $f(1, 0) = -3$.

7. (B2) This problem has two parts.

(a) A rectangular pyramid is designed with the constraint that the **sum** of the **area of its rectangular base** (xy) with **the sum of the areas of its orthogonal triangle cross sections** ($\frac{xz}{2}$ **and** $\frac{yz}{2}$) cannot exceed 12 square meters. Find the dimensions x, y, and z that yield the maximum volume using the method of Lagrange multipliers. Note: you may assume a maximum exists. Hint: Maybe multiply your constraint equation by 2 before you set up Lagrange multipliers to avoid fractions.



Note: The volume of a rectangular pyramid is $V = \dfrac{xyz}{3}$.

Solution: We want to maximize $V = \frac{xyz}{3}$ subject to:

$$12 = xy + \frac{xz}{2} + \frac{yz}{2} \implies 24 = 2xy + xz + yz$$

We may assume all lengths are positive. The Lagrange system is:

$$\begin{aligned} \tfrac{yz}{3} &= \lambda\,(2y+z) \\ \tfrac{xz}{3} &= \lambda\,(2x+z) \\ \tfrac{xy}{3} &= \lambda\,(x+y) \end{aligned}$$

Divide the first two equations to find:

$$\frac{y}{x} = \frac{2y+z}{2x+z} \implies 2xy + yz = 2xy + xz \implies yz = xz \implies x = y$$

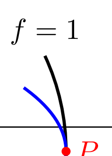
Divide the first equation by the last equation to find:

$$\frac{z}{x} = \frac{2y+z}{x+y} \implies xz + yz = 2xy + xz \implies yz = 2xy \implies z = 2x$$

Writing our constraint equation in terms of x we get:

$$24 = 2x(x) + x(2x) + x(2x) = 6x^2 \implies x = 2 \text{ m} \implies y = 2 \text{ m} \implies z = 4 \text{ m}$$

(b) The point P is on both the level curves $f = 1$ and $g = 1$. Is it **possible** according to Lagrange multipliers that P is an extremizer of f subject to $g = 1$? Briefly explain. Note: an answer without correct explanation receives no credit. Note: you may assume the gradients of both functions at P exist and are nonzero.

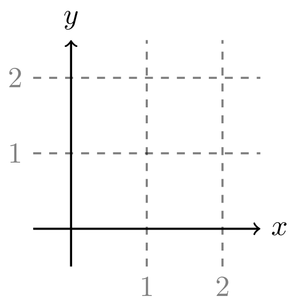


Solution: It is possible because the gradient $\nabla f(P)$ is parallel to $\nabla g(P)$. In other words the Lagrange equation $\nabla f(P) = \lambda \nabla g(P)$ has a solution λ.

8. (B3) (4 points) This problem has multiple parts.

(a) Find the integral. Hint: the given order of integration is no good.

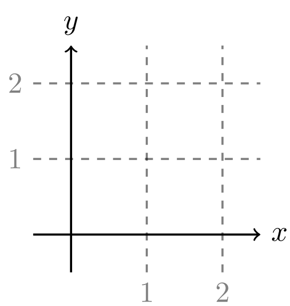
$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\ dydx$$



Solution: In the other order of integration we have $0 \leq x \leq y^2$ and $0 \leq y \leq 1$.

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\ dydx = \int_0^1 \int_0^{y^2} e^{y^3}\ dxdy = \int_0^1 y^2 e^{y^3}\ dx = \left.\frac{e^{y^3}}{3}\right|_{y=0}^1 = \frac{e}{3} - \frac{1}{3}$$

(b) Let D be the region in the xy–plane bounded by the lines $y = x$ and $y = 2 - x$ and $x = 0$. Electric charge is distributed over D so its charge density (in coulombs per square meter) is $\sigma(x, y) = 6x$. Find the total charge on D.



Solution: Our region of integration is described by $x \leq y \leq 2 - x$ and $0 \leq x \leq 1$ (the upper bound $x = 1$ is because $y = x$ and $y = 2 - x$ interesct when $x = 1$). We set up:

$$\int_0^1 \int_x^{2-x} 6x\ dydx = \int_0^1 6x(2 - x) - 6x(x)\ dx = \int_0^1 12x - 12x^2\ dx = 6 - 4 = 2 \text{ coulombs}$$

9. (B4) (4 points) This problem has multiple parts

(a) Let E be the region bounded by the downwards paraboloid $z = 13 - x^2 - 2y^2$ and the upwards paraboloid $z = 1 + 2x^2 + y^2$. Set up the integral over E in the order $dzdxdy$.

$$\iiint_E f(x, y, z)\ dV = \int_?^? \int_?^? \int_?^? f(x, y, z)\ dzdydx$$

Hint: find the shadow of this region in the xy-plane by considering where the surfaces intersect: it should be a familiar region. You also have to find bounds in order $dydx$ for this familiar region: make sure you are **not** using polar! x, y, and z only.

Solution: The surfaces intersect when:

$$13 - x^2 - 2y^2 = 1 + 2x^2 + y^2 \implies 12 = 3x^2 + 3y^2 \implies 4 = x^2 + y^2$$

This is the circle of radius 2 centered at the origin. The shadow is the disk with this circle as its edge.

So we get bounds:

$$1 + 2x^2 + y^2 \le z \le 13 - x^2 - 2y^2$$
$$-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$$
$$-2 \le x \le 2$$

(b) Find the **volume** of the region in the first octant $(x, y, z \ge 0)$ bounded by the surfaces $y = \sqrt{1 - x}$ and $z = 2y$ by setting up **and computing** a triple integral in the order $dzdydx$. Note: setting up the bounds in this integral should be extremely quick in comparison to the previous one.

Solution: The bounds are:

$$0 \le z \le 2y$$
$$0 \le y \le \sqrt{1 - x}$$
$$0 \le x \le 1$$

So the volume is:

$$\int_0^1 \int_0^{\sqrt{1-x}} \int_0^{2y} 1\ dzdydx = \int_0^1 \int_0^{\sqrt{1-x}} 2y\ dydx = \int_0^1 1 - x\ dx = 1 - \frac{1}{2} = \frac{1}{2}$$

10. (B5) (4 points) This problem has multiple parts.

(a) Use cylindrical coordinates to find:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} 8z \; dzdydx$$

Hint: maybe try $dzdrd\theta$. What does the shadow in the xy-plane look like?

Solution: The xy-shadow is the quarter of the disk $x^2 + y^2 \leq 1$ in the first quadrant, and so gives bounds on r and θ as $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$.

The bounds on z are:

$$0 \leq z \leq \sqrt{x^2+y^2} \implies 0 \leq z \leq r$$

We are now ready to compute:

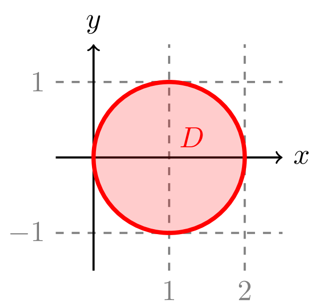
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} 8z \; dzdydx = \int_0^{\pi/2} \int_0^1 \int_0^r 8z \; rdzdrd\theta = \int_0^{\pi/2} \int_0^1 4r^3 \; drd\theta = \int_0^{\pi/2} 1 \; d\theta = \pi/2$$

(b) Let D be the shifted–right–one unit–disk $(x-1)^2 + y^2 \leq 1$.

Set up **but do not compute** the following integral using polar coordinates in order $drd\theta$.

$$\iint_D x \; dA$$

Hint: rewrite the equation $(x-1)^2 + y^2 = 1$ in polar and solve for r. The picture can help with bounds on θ.



Solution: Rewrite the equation for the edge of D as:

$$x^2 - 2x + 1 + y^2 = 1 \implies x^2 + y^2 = 2x \implies r^2 = 2r\cos\theta \implies r = 2\cos\theta$$

So the bounds on r are $0 \leq r \leq 2\cos\theta$. From the picture: the bounds on θ are $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Our integral is:

$$\iint_D x \; dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\cos\theta \; rdrd\theta$$

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