1. (A1) (4 points) This problem has multiple parts.

(a) Suppose $\mathbf{v}$ and $\mathbf{w}$ are 3D vectors that form a parallelogram with area 4. Use only this information to find:

$$\Big((2\mathbf{v}+3\mathbf{w})\times\mathbf{w}\Big)\cdot(\mathbf{v}\times\mathbf{w})$$

Hint: start by simplifying the expression in big parentheses: $(2\mathbf{v}+3\mathbf{w})\times\mathbf{w}$. Later on: remember that a vector dotted with itself can be written another way...also: what is the length of $\mathbf{v}\times\mathbf{w}$ based on the given info?

Solution: First we find:

$$(2\mathbf{v}+3\mathbf{w})\times\mathbf{w} = 2(\mathbf{v}\times\mathbf{w}) + 3(\mathbf{w}\times\mathbf{w}) = 2(\mathbf{v}\times\mathbf{w})$$

so:

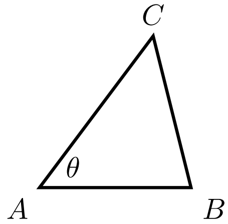
$$\Big((2\mathbf{v}+3\mathbf{w})\times\mathbf{w}\Big)\cdot(\mathbf{v}\times\mathbf{w}) = 2\,(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w}) = 2\|\mathbf{v}\times\mathbf{w}\|^2 = 2\cdot 4^2 = 32$$

(b) Find the the **scalar** (not vector) component of $\mathbf{w}=\langle 1,1,1\rangle$ along $\mathbf{v}=\langle 2,1,3\rangle$, i.e. find $\text{comp}_{\mathbf{v}}(\mathbf{w})$.

Solution: We find:

$$\text{comp}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|} = \frac{6}{\sqrt{14}}$$

(c) Find the angle θ at point A in the triangle with vertices $A=(1,1,1)$, $B=(2,3,1)$, $C=(1,2,4)$. You may leave your answer unsimplified. Note: image below not to scale



Solution: We calculate:

$$\mathbf{AB} = \langle 1,2,0\rangle \text{ and } \mathbf{AC} = \langle 0,1,3\rangle$$

and compute:

$$\mathbf{AB}\cdot\mathbf{AC} = \|\mathbf{AB}\|\|\mathbf{AC}\|\cos\theta \implies 2 = \sqrt{5}\cdot\sqrt{10}\cos\theta \implies \cos^{-1}\left(\frac{2}{\sqrt{50}}\right) = \theta$$

2. (A2) (4 points) Let $\mathcal{P}_1$ be the plane both containing the point $P(1,3,2)$ and parallel to the plane:

$$\mathcal{P}_2 : 2x + 3y - z = 4$$

(a) Find an equation equivalent to one of form $ax + by + cz = d$ for $\mathcal{P}_1$.

Solution: We get:

$$2(x-1) + 3(y-3) - (z-2) = 0$$

(b) Find the distance between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

Solution: A common normal vector to the planes is $\mathbf{n} = \langle 2, 3, -1 \rangle$. We are given a point $P(1,3,2)$ on first plane. A point on the second plane $\mathcal{P}_2$ is $Q(2,0,0)$. The distance is:

$$\left| \frac{\mathbf{n} \cdot (\mathbf{q} - \mathbf{p})}{\|\mathbf{n}\|} \right| = \left| \frac{\langle 2,3,-1 \rangle \cdot \langle 1,-3,-2 \rangle}{\sqrt{14}} \right| = \frac{5}{\sqrt{14}}$$

(c) Let ℓ be the line through $P(1,3,2)$ and which is orthogonal to both planes.

(i) Parametrize ℓ and call the parameter t. Hint: what direction vector would make the line orthogonal (perpendicular) to the planes?

(ii) Find the value of your parameter t at which this ℓ intersects the plane $\mathcal{P}_2$. Hint: plug in.

Solution: A parametrization for this line is $\langle x, y, z \rangle = \langle 1 + 2t, 3 + 3t, 2 - t \rangle$ which we subsitute into the $\mathcal{P}_2$ equation to get:

$$2(1+2t) + 3(3+3t) - (2-t) = 4 \implies 9 + 14t = 4 \implies t = -\frac{5}{14}$$

3. (A3) (4 points) This problem has multiple parts.

(a) Find the path $\mathbf{r}_1(t)$ so that $\mathbf{r}_1(0) = \langle 0, 4, 2\rangle$ and $\mathbf{r}_1'(t) = \langle \cos t, 2t+1, e^t\rangle$.

Hint: some of the derivatives $(\sin t)' = \cos t$ and $(\cos t)' = -\sin t$ and $(e^t)' = e^t$ might help.

Solution: We find:

$$\mathbf{r}_1(t) = \langle \sin t, t^2 + t + 4, e^t + 1\rangle$$

(b) Name the surface $\mathcal{S}$ defined by $4y = x^2 + 4z^2$. No work required.

paraboloid saddle ellipsoid/sphere 1-sheeted hyperboloid 2-sheeted hyperboloid cone

Solution: paraboloid

(c) Suppose $\mathbf{r}(t)$ is a path so $\|\mathbf{r}(t)\|$ is constant. Show that $\mathbf{r}(t) \perp \mathbf{r}'(t)$.

Hint: the given info tells you what $\|\mathbf{r}(t)\|^2$ equals, what is it? Write $\|\mathbf{r}(t)\|^2$ in terms of dot products.

Solution: Since $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$ is constant we have:

$$\Big(\mathbf{r}(t)\cdot\mathbf{r}(t)\Big)' = 0 \implies \mathbf{r}(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}(t) = 0 \implies 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0 \implies \mathbf{r}(t)\perp\mathbf{r}'(t)$$

4. (A4) (4 points) For this problem consider the function $f(x, y) = \sin(2x + x^2 + 3y)$.

(a) Find an equation of the tangent plane to the graph of f at $P(0, 0, 0)$.

Solution: We find:

$$f_x(0,0) = \cos\Big(2(0) + (0)^2 + 3(0)\Big) \cdot \Big(2 + 2(0)\Big) = 2$$

$$f_y(0,0) = \cos\Big(2(0) + (0)^2 + 3(0)\Big) \cdot (3) = 3$$

An equation of the tangent plane is:

$$z = 2x + 3y$$

(b) If placed starting at P, does the vector $\mathbf{v} = \langle 1, 1, 5 \rangle$ point tangent to the graph of f? Hint: first find a normal vector to the tangent plane at P with the help of previous work. What should the relationship be between a tangent vector and a normal vector? Check whether this relation holds!

Solution: A normal vector to the tangent plane at P is $\mathbf{n} = \langle -2, -3, 1 \rangle$. We compute:

$$\mathbf{n} \cdot \mathbf{v} = \langle -2, -3, 1 \rangle \cdot \langle 1, 1, 5 \rangle = 0$$

So $\mathbf{v}$ is tangent to the surface (it is orthogonal to $\mathbf{n}$).

(c) Find g_{yxyx} for the following function. Hint: order.

$$g(x, y, z) = x \sin(ye^{ye^y}) + x^2y^3$$

Solution: There are two x-derivatives which kill the first term. The last term, after two x-derivatives and two y-derviatives:

$$x^2y^3 \xrightarrow{\text{derivatives}} 12y$$

Hence:

$$g_{yxyx}(x, y) = 12y$$

5. (A5) (4 points) For this problem you are given that $f(x, y, z) = xyz$ and $\mathbf{r}(t)$ is a path so:

$$\mathbf{r}(3) = \langle 1, 2, 2\rangle$$

$$\mathbf{r}'(3) = \langle 1, 1, 1\rangle$$

(a) If $g(t) = f(\mathbf{r}(t))$ then find $g'(3)$.

Solution: We have:

$$g'(3) = \nabla f(\mathbf{r}(3)) \cdot \mathbf{r}'(3) = \langle yz, xz, xy\rangle|_{x=1,y=2,z=2} \cdot \langle 1, 1, 1\rangle = \langle 4, 2, 2\rangle \cdot \langle 1, 1, 1\rangle = 8$$

(b) Find an equation for the tangent plane to the level surface $f = 6$ (aka the surface $xyz = 6$) at $P(1, 2, 3)$.

Solution: We find $\nabla f(1, 2, 3) = \langle 6, 3, 2\rangle$ and so an equation for the tangent plane is:

$$6(x - 1) + 3(y - 2) + 2(z - 3) = 0$$

(c) Find the **unit** direction $\mathbf{u}$ (a **unit vector**) starting from $P(1, 2, 3)$ in which the rate of change of f is **largest**. Calculate also the directional derivative (a **number**) in this direction.

Solution: The unit vector in question is:

$$\mathbf{u} = \frac{\nabla f(P)}{\|\nabla f(P)\|} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle$$

The directional derivative in this direction is:

$$\|\nabla f(P)\| = 7$$

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