Example 1. Let $\mathbf{v}$ and $\mathbf{w}$ be unit vectors so that $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}$ and find:

$$\|2\mathbf{v} + \mathbf{w}\|^2$$

Remember a unit vector is a vector with length 1.

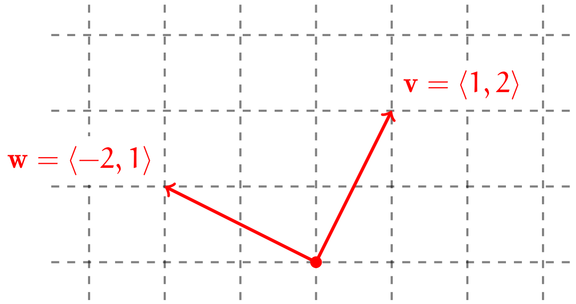
Also remember: $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

The new dot product properties:
distributivity: $\mathbf{v} \cdot (\mathbf{w} + \mathbf{r}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{r}$
and: $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{r} = \mathbf{v} \cdot \mathbf{r} + \mathbf{w} \cdot \mathbf{r}$
commutativity: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
and: $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$

Distributivity is all about distributing multiplication over addition. Commutativity is new, and it tells us we can change the order, in this case of multiplication, without changing the result. It's not like we'll ever encounter any multiplication that is not commutative. Right...right professor?

A. **Dot Products and Angles.** Can the dot product of two nonzero vectors be 0?

Would that be scary?

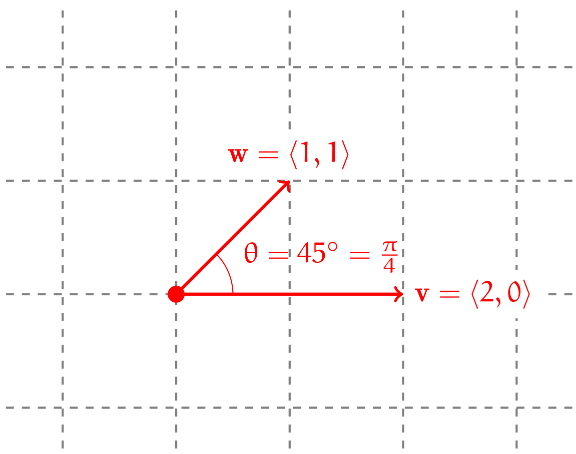


> Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** when:
>
> $\mathbf{v} \cdot \mathbf{w} =$
>
> in which case we write $\mathbf{v} \perp \mathbf{w}$.

Orthogonal and perpendicular mean the same thing. Tomato tomato.

There is an even more general relationship between dot products and angles. Consider two vectors and the angle between them.

The angle could be the short way around as depicted, or the long way around, but it don't matter. What follows is still true.



For $\mathbf{v} = \langle 1, 1 \rangle$ and $\mathbf{w} = \langle 2, 0 \rangle$ let us compute:

$$\mathbf{v} \cdot \mathbf{w} =$$

$$\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta =$$

> If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$ then:
>
> $\mathbf{v} \cdot \mathbf{w} =$
>
> which agrees nicely with the fact that $\mathbf{v} \cdot \mathbf{w} = 0$ when $\theta =$

We have not even attempted to really establish why this is always true. If you are curious why, ask your TA or professor in office hours. We can at least cover a basic case: when one of the vectors is parallel to the x-axis.

Example 2. Find all vectors orthogonal to:

$$\mathbf{v} = \langle 3, 4 \rangle$$

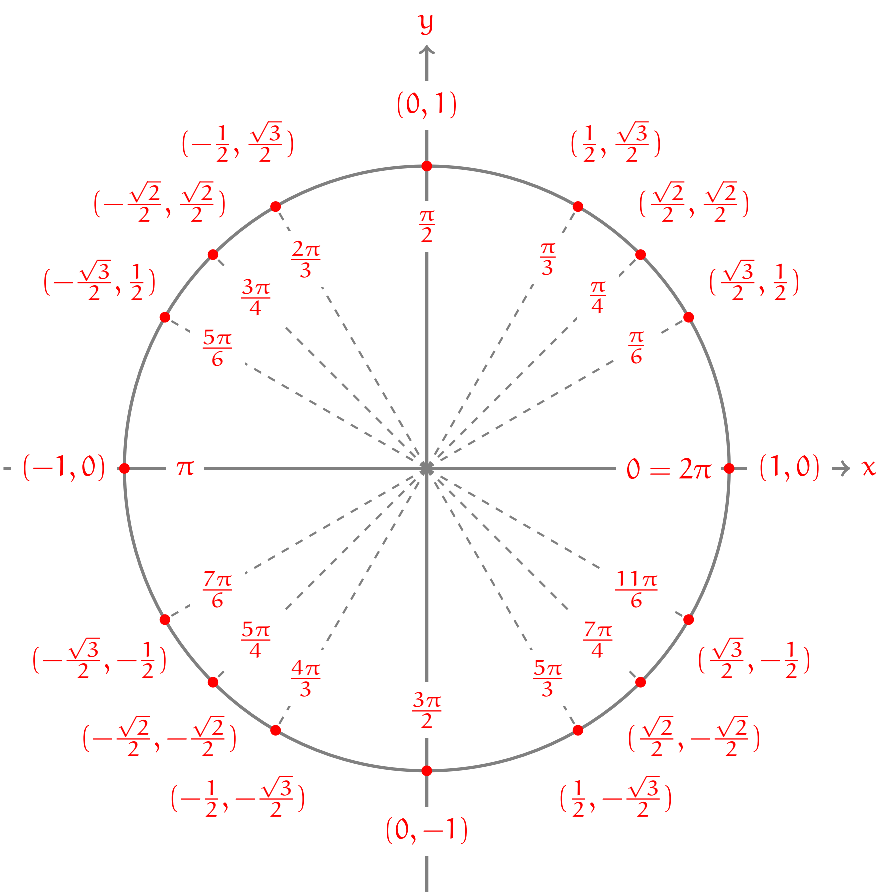
that have the same length as $\mathbf{v}$.

The rotation of vector $\langle a, b \rangle$ counterclockwise by $90°$ is $\langle -b, a \rangle$ while its rotation clockwise by $90°$ is $\langle b, -a \rangle$

Example 3. Find the angle between the vectors:

$$\mathbf{v} = \langle 1, 0, 1 \rangle \text{ and } \mathbf{w} = \langle -1, 1, 0 \rangle$$

You may find the **unit circle** helpful for locating the correct value.



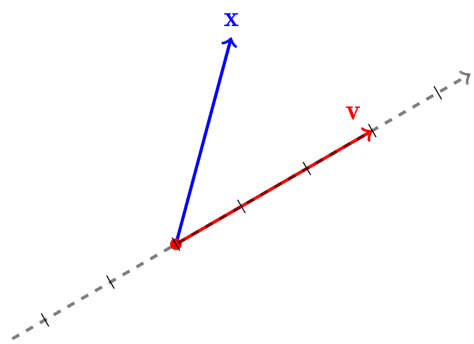
If θ tells you the angle of counterclockwise rotation from the positive x-axis to the position vector of the point, then remember that $\cos\theta$ is the x-coordinate of the point and $\sin\theta$ is the y-coordinate.

The unit circle also helps us see that if $\mathbf{v} \cdot \mathbf{w}$ is negative, then the angle between the vectors is larger that $90°$. Or more precisely, a counterclockwise rotation of one vector to another would have to be by an angle of magnitude greater than $90°$.

B. **Projections.** Consider a nonzero vector $\mathbf{v}$.

Any vector $\mathbf{x}$ has a unique **orthogonal decomposition** as a sum of a vector that is **parallel** to $\mathbf{v}$ with a vector that is **orthogonal** to $\mathbf{v}$.

Remember that $\mathbf{x}$ being parallel to $\mathbf{v}$ means it is a scalar multiple of $\mathbf{v}$. And $\mathbf{x}$ being orthogonal to $\mathbf{v}$ means $\mathbf{x} \cdot \mathbf{v} = 0$



We might call the parallel part in this orthogonal decomposition the **parallel component** in which case I guess the other part would be **orthogonal component**.

The **scalar component of $\mathbf{x}$ along $\mathbf{v}$** is the tickmark along this dashed axis marking the parallel component. It is denoted by $\text{comp}_{\mathbf{v}}(\mathbf{x})$. Let us derive a formula.

Remember that a scalar is a real number. So the scalar component is a real number. When the scalar component is negative, the parallel component is opposite the direction of $\mathbf{v}$!

$$\text{comp}_{\mathbf{v}}(\mathbf{x}) =$$

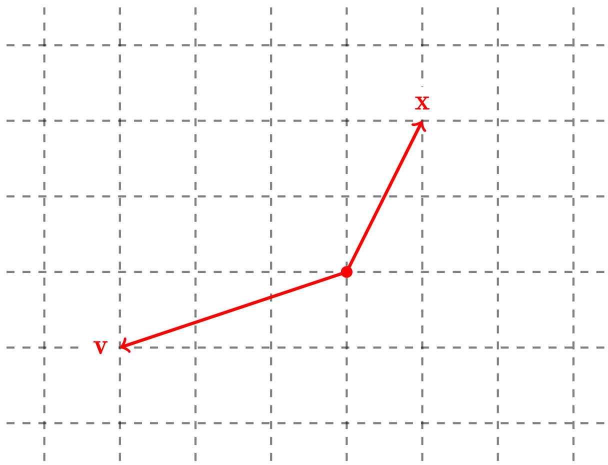
Another name for the parallel component in this decomposition is the **(orthogonal) projection of $\mathbf{x}$ onto $\mathbf{v}$** and is denoted by $\text{proj}_{\mathbf{v}}(\mathbf{x})$. Let us derive a formula.

The othogonal projection is a vector. The scalar component is a scalar.

$$\text{proj}_{\mathbf{v}}(\mathbf{x}) =$$

To memorize a formula you should know where it comes from and what it means. Do not try to force facts into your brain without context. Foundation makes everything more secure.

Example 4. Find the orthogonal projection of $\mathbf{x} = \langle 1, 2 \rangle$ onto $\mathbf{v} = \langle -3, -1 \rangle$ and find the scalar component of $\mathbf{x}$ along $\mathbf{v}$.



What is the orthogonal projection of me onto my neighbor?