A. **Cross Products.** Let us talk about a second useful way to multiply vectors. We call it the cross product.

$$(\text{3D vector}) \times (\text{3D vector}) = (\text{3D vector})$$

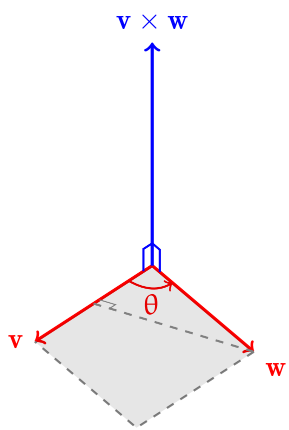
Let us explain what characterizes it.

Cross products are only for 3D vectors? I wonder why?

By "smaller angle" we mean an angle in the range $0 \leqslant \theta \leqslant \pi$.

If $\mathbf{v}$ and $\mathbf{w}$ are 3D vectors with smaller angle θ between them, then their cross product $\mathbf{v} \times \mathbf{w}$ is the 3D vector that:

- is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$
- has direction determined by the **righthand rule**



In words, to execute the righthand rule, curl your fingers in the direction of shortest rotation from $\mathbf{v}$ to $\mathbf{w}$, in which case your thumb is pointing in the direction of the cross product.

- has length equal to the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$, i.e.:

$$\|\mathbf{v} \times \mathbf{w}\| =$$

For any 3D vectors $\mathbf{v}$ and $\mathbf{w}$ we have the following properties.

Anti–Commutativity: $\mathbf{w} \times \mathbf{v} =$

Self–Annihilating: $\mathbf{v} \times \mathbf{v} =$

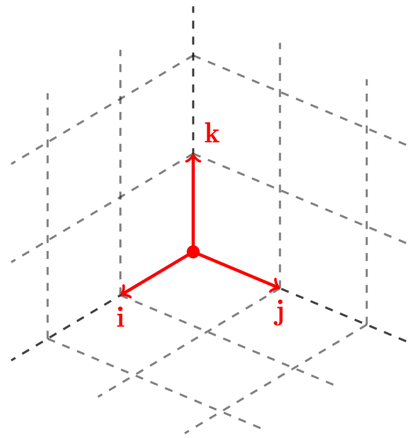
Oh dear lord I cannot just change the order of mulitplication like I have been doing my entire dang life?

Example 1. Compute the cross products involving the special vectors:

$\mathbf{i} =$

$\mathbf{j} =$

$\mathbf{k} =$



$\mathbf{i}, \mathbf{j}, \mathbf{k}$ is really physics notation. Mathematicians might prefer $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

$\mathbf{i} \times \mathbf{j} =$

$\mathbf{i} \times \mathbf{k} =$

$\mathbf{j} \times \mathbf{k} =$

$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) =$

$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} =$

Oh my god. You **cannot** freely move parentheses around? That's SO messed up. This is referred to as the **failure** of associativity: $(\mathbf{v} \times \mathbf{w}) \times \mathbf{r} \neq \mathbf{v} \times (\mathbf{w} \times \mathbf{r})$. Associativity is all about moving parentheses around.

Next use the idea that any 3D vec can be written in terms of these special vecs:

$\langle a, b, c \rangle =$

along with the new properties in the margin to find:

$\langle 1, 2, 0 \rangle \times \langle 2, 0, 0 \rangle$

distributivity:
$(\mathbf{v} + \mathbf{w}) \times \mathbf{r} = \mathbf{v} \times \mathbf{r} + \mathbf{w} \times \mathbf{r}$
$\mathbf{v} \times (\mathbf{w} + \mathbf{r}) = \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{r}$
commutativity with scalars:
$(c\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (c\mathbf{w}) = c(\mathbf{v} \times \mathbf{w})$

B. **Computing Cross Products.** So far cross-products seem tough to compute.

Is there not a magic formula?

The cross–product $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$ equals the **determinant**:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which means it equals:

$$+ \begin{vmatrix} \mathbf{i} & & \\ & a_2 & a_3 \\ & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} & \mathbf{j} & \\ a_1 & & a_3 \\ b_1 & & b_3 \end{vmatrix} + \begin{vmatrix} & & \mathbf{k} \\ a_1 & a_2 & \\ b_1 & b_2 & \end{vmatrix}$$

We call a rectangular array of entries a matrix. The determinant of a 2 by 2 matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and this is what we are computing thrice as part of calculating the cross product.

Find a nonzero vector orthogonal to both $\mathbf{v} = \langle 2, -1, 1 \rangle$ and $\mathbf{w} = \langle -3, -1, 2 \rangle$.

From now on, if you ever need a vector orthogonal to two other 3D vectors, cross products better leap into your mind!

C. **Scalar Triple Product.** The cross product and dot product do not have to operate in isolation. We can execute them in succession:

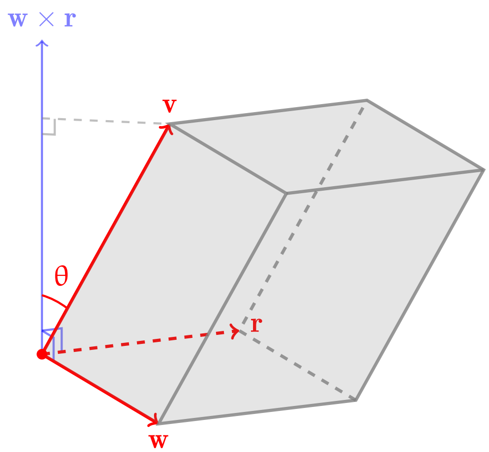
The **scalar triple product** of 3D vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{r}$ is:

$$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{r}) = \langle v_1, v_2, v_3 \rangle \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \begin{vmatrix} & & \\ w_1 & w_2 & w_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

Underlying the last equality is that:
$\mathbf{v} \cdot \mathbf{i} = \langle v_1, v_2, v_3 \rangle \cdot \langle 1, 0, 0 \rangle = v_1$
$\mathbf{v} \cdot \mathbf{j} = \langle v_1, v_2, v_3 \rangle \cdot \langle 0, 1, 0 \rangle = v_2$
$\mathbf{v} \cdot \mathbf{k} = \langle v_1, v_2, v_3 \rangle \cdot \langle 0, 0, 1 \rangle = v_3$

It has an important geometric meaning. Each pair of vectors from $\mathbf{v}$, $\mathbf{w}$, $\mathbf{r}$ forms a parallelogram, and together they form an object called a **parallelepiped**.

Say that three times fast. You can think of it as a slanted cube.



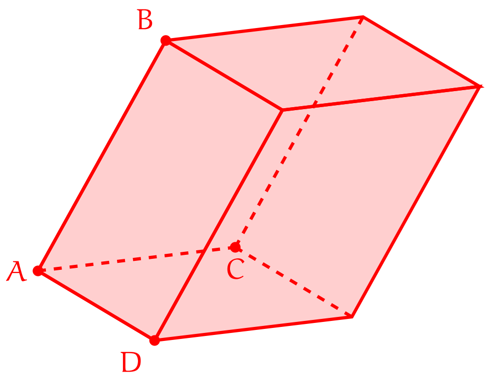
Then:

$$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{r}) =$$

The parallelepiped formed by $\mathbf{v}$, $\mathbf{w}$, $\mathbf{r}$ has **signed volume** equal to their scalar triple product.

A signed volume can be negative, specifically in this case if the shortest angle between $\mathbf{v}$ and $\mathbf{w} \times \mathbf{r}$ is more than $90°$ in magnitude.

Example 2. Find the volume of the parallelepiped with vertex $A(1, 0, 3)$ adjacent to vertices $B(2, 2, 6)$, $C(5, 5, 9)$, $D(8, 8, 13)$.



This is probably not even close to how this parallelepiped actually looks in xyz-space. Nonetheless we make a sketch because it helps organize our thoughts. And god knows I need help with that.