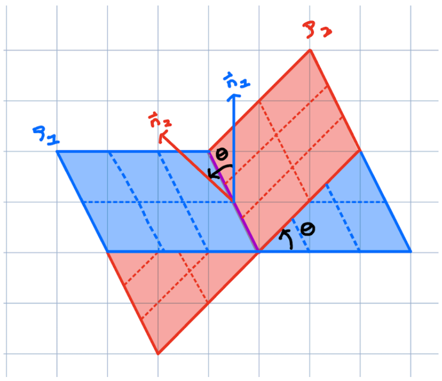
Example 1. Consider the planes:

$$\mathcal{P}_1 : 2x + y + 4z = 2$$

$$\mathcal{P}_2 : 2x - 4y + z = 2$$

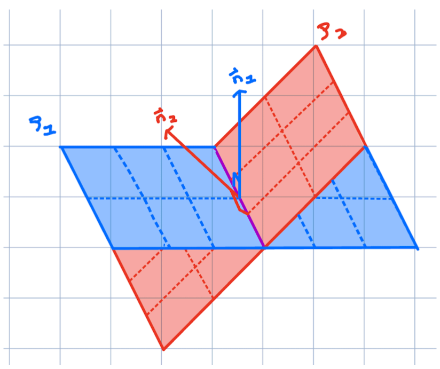
(a) Find the angle between normal vectors to the planes.



Typically we consider this angle (between the normal vectors) to be the angle between the planes, because it effectively captures how one is tilted with respect to the other. Therefore if you are asked for the angle between two planes, know that you are really being asked for the angle between their normal vectors.

(b) Parametrize the line of intersection of the two planes.

Hint: Note that $P(1, 0, 0)$ is on both planes.

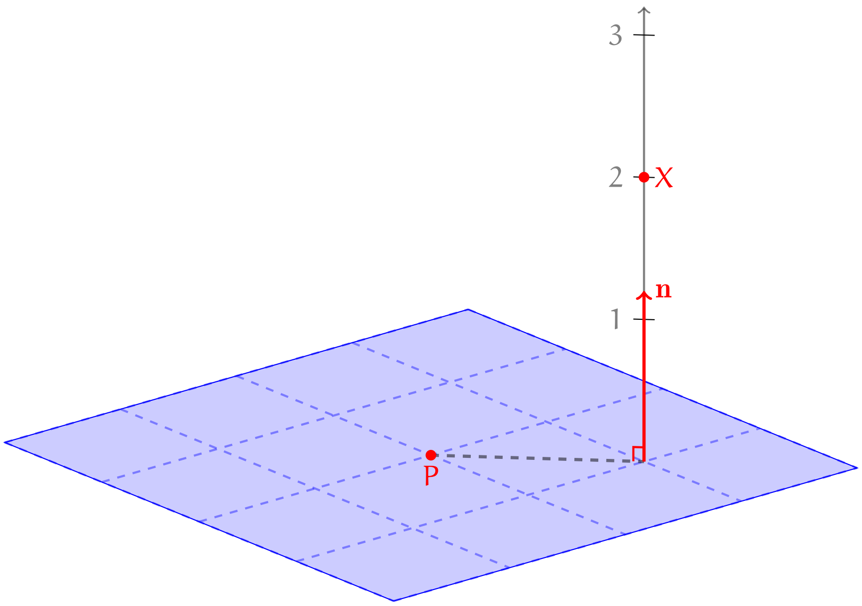


A. **Distances.** We can chat about distances between lines, planes, and points. Let us begin with the distance between a point X and a plane $\mathcal{P}$.

Precisely: the **distance** will always refer to the **shortest** distance between the objects.

Say we know a point P on the plane and a normal vector $\mathbf{n}$ to the plane.

We have to know something to do something.



The distance between point X and plane $\mathcal{P}$ is:

where P is a known point on the plane, and $\mathbf{n}$ is a known normal vector.

In this formula remember that P is a point on the plane and $\mathbf{n}$ is a normal vector to the plane.

Example 2. Consider the planes:

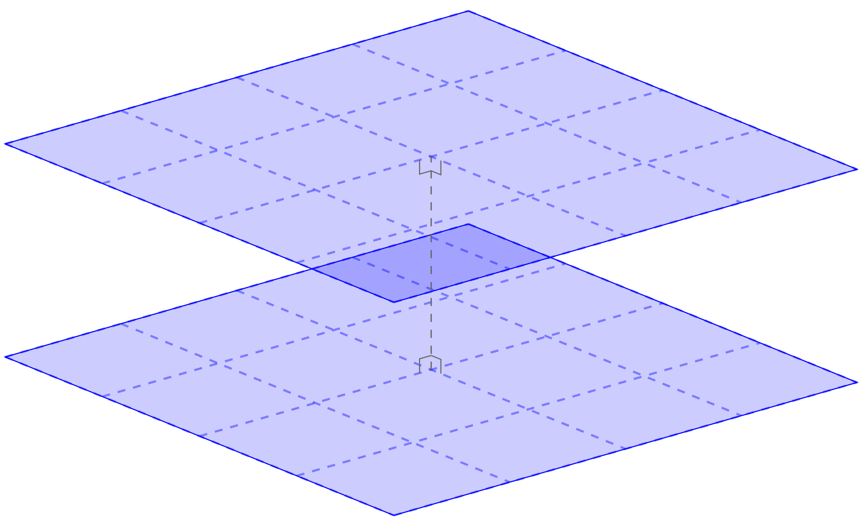
$$\mathcal{P}_1 : 10x + 2y - 2z = 6$$

$$\mathcal{P}_2 : 5x + y - z = 1$$

(a) Explain why these planes are parallel.

Planes are parallel if their normal vectors are parallel!

(b) Knowing that they are parallel, find the distance between them.

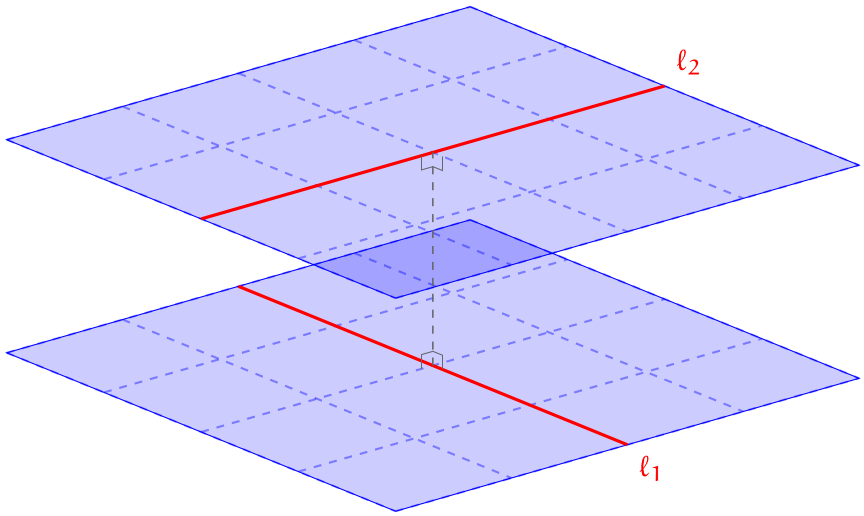


We have derived a nice formula. Two parallel planes will have a common normal $\mathbf{n}$ and so will have scalar equations of form $\mathbf{n} \cdot \mathbf{x} = d_1$ and $\mathbf{n} \cdot \mathbf{x} = d_2$, in which case the distance between the planes is:

$$\frac{|d_2 - d_1|}{\|\mathbf{n}\|}$$

Example 3. Two lines are **skew** if they are not parallel and not intersecting.

When we say that lines are not parallel, we mean that their direction vectors are not parallel, which means their direction vectors are not scalar multiples of each other.



Consider the lines with parametrizations:

$$\ell_1 : \mathbf{r}_1(t) = \langle 1+t, -2+3t, 4-t\rangle$$

$$\ell_2 : \mathbf{r}_2(t) = \langle 2t, 3+t, -1+4t\rangle$$

(a) Assess whether the lines are parallel to each other.

(b) Find the distance between the two lines.

Any two lines can be situated in parallel planes, as we can see by using a cross product to find a vector that is normal to both lines. When the lines are not parallel to each other, the distance between those planes is the same as the distance between the lines.