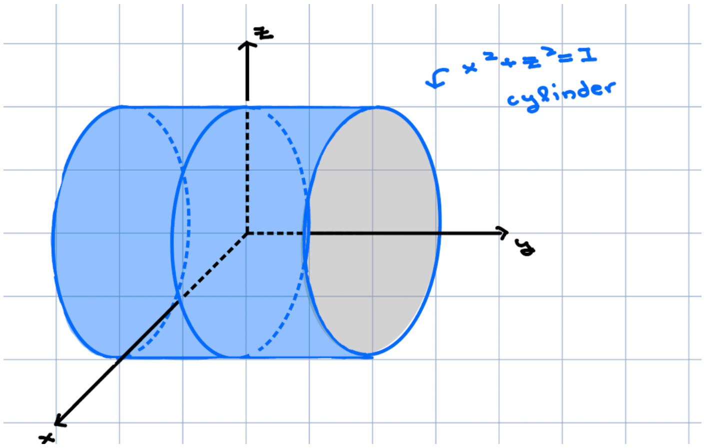
A. **Cylinders and Slices.** Planes are examples of **surfaces**, specifically they are surfaces defined by an equation involving a linear function of two variables. A **quadric surface** is defined by an equation involving a degree 2 polynomial of two variables.

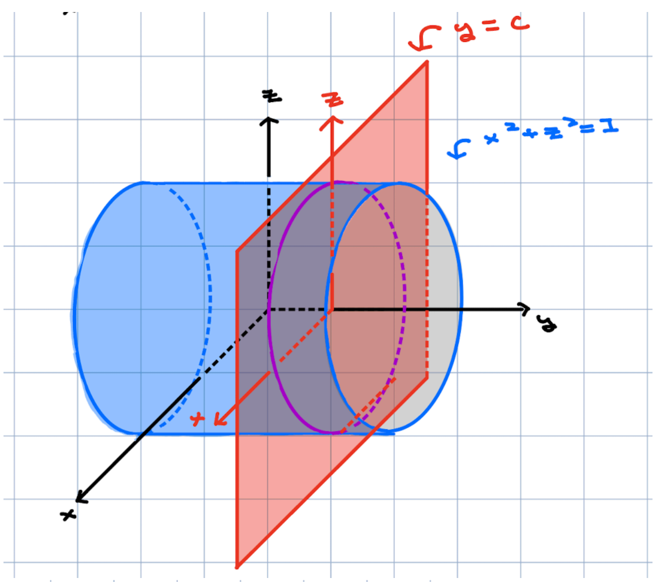
A **polynomial** is a sum of terms, each of which looks like a scalar times some variables, like $2x + 3xy + 5x^2y$. The **degree** is related to the maximum number of variables that are multiplied together. For example the above polynomial has degree 3 because of the term $5x^2y = 5xxy$ which has 3 variables multiplied together.

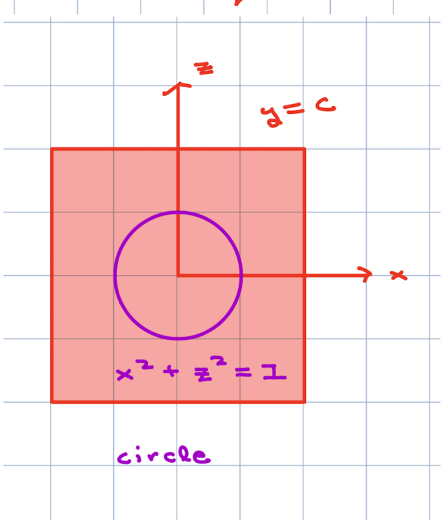
For example the equation $x^2 + z^2 = 1$ defines the circular cylinder:



Remember, an equation defines a surface by giving you a rule to assess whether or not a point (x, y, z) is on the surface. In this case the rule is $x^2 + z^2 = 1$. Hence $(1, 2, 0)$ is on the surface, because $1^2 + 0^2 = 1$, but $(1, 2, 1)$ is not on the surface, because $1^2 + 1^2 \neq 1$.

We could reconstruct its graph by looking at its $y = c$ **slices**, where c is a constant.

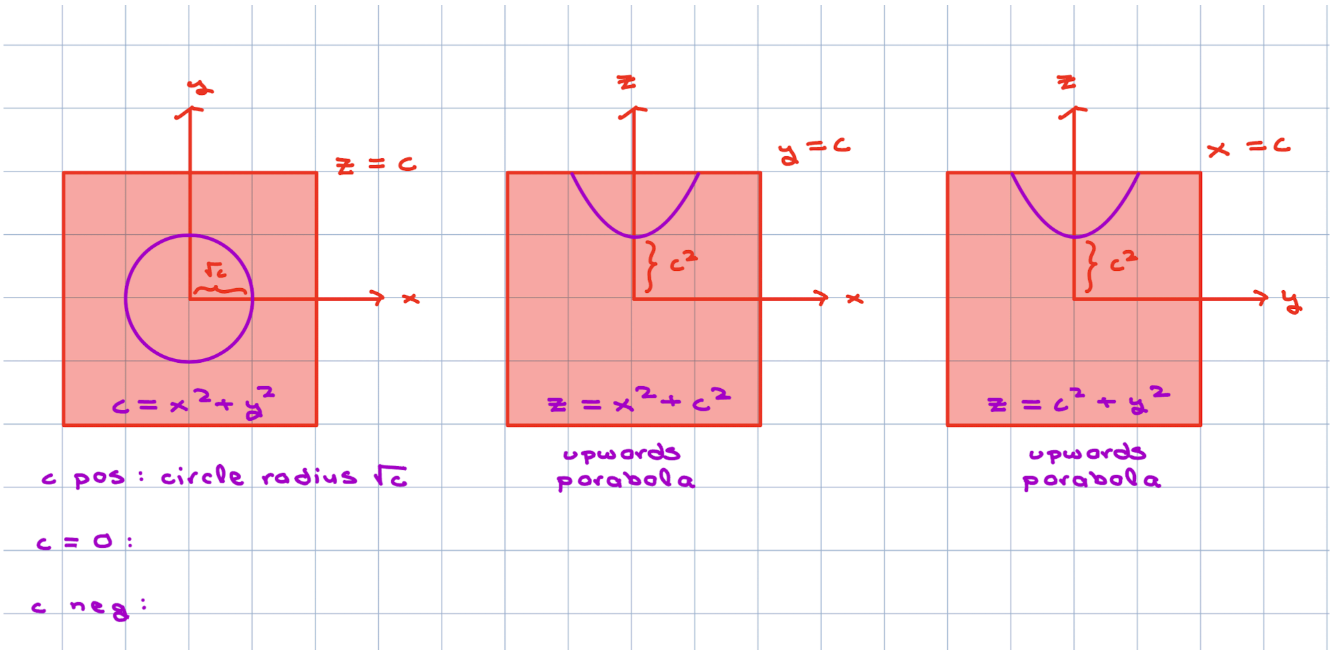




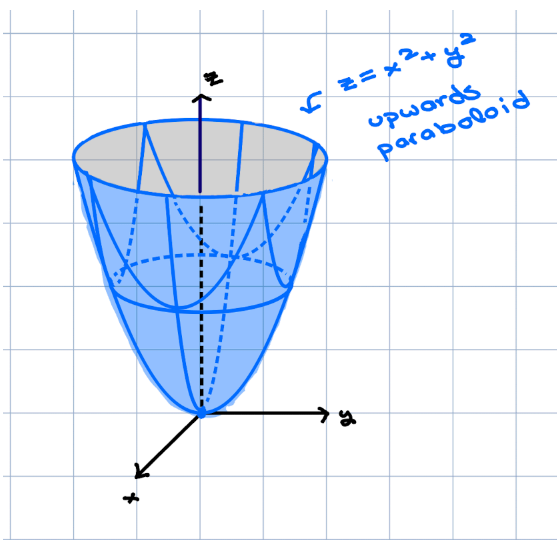
In this case, the equation of the $y = c$ slice is $x^2 + z^2 = 1$, independent of the constant c. Therefore this surface has **y-translational symmetry**, meaning you can take any point on the surface, then translate it in the y-direction (in the picture this would be left/right), and you will still be on the surface!

A **[your-favorite-curve] cylinder** is a surface obtained from taking **[your-favorite-curve]** and translating it in one direction. Hence this cylinder is a **circular** cylinder. But you could have a **parabolic**cylinder (translate a parablola) or an **elliptic** cylinder (translate an ellipse) and so on. See discussion for details.

B. **Paraboloids.** We next look at the equation $z = x^2 + y^2$ and consider its $z = c$, $y = c$, and $x = c$ slices.

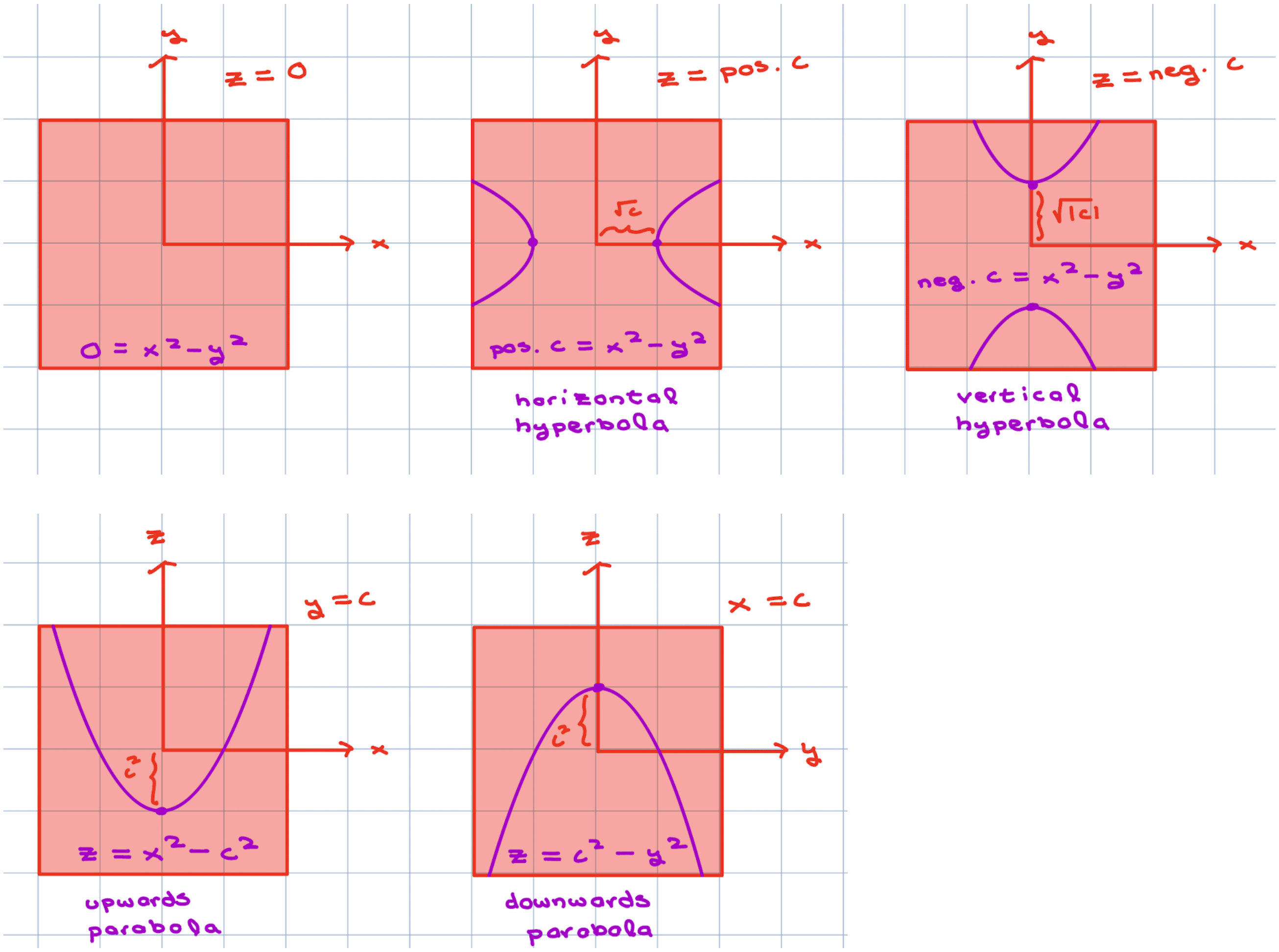


We put these together to obtain the surface it defines, which we call an upwards **paraboloid**.

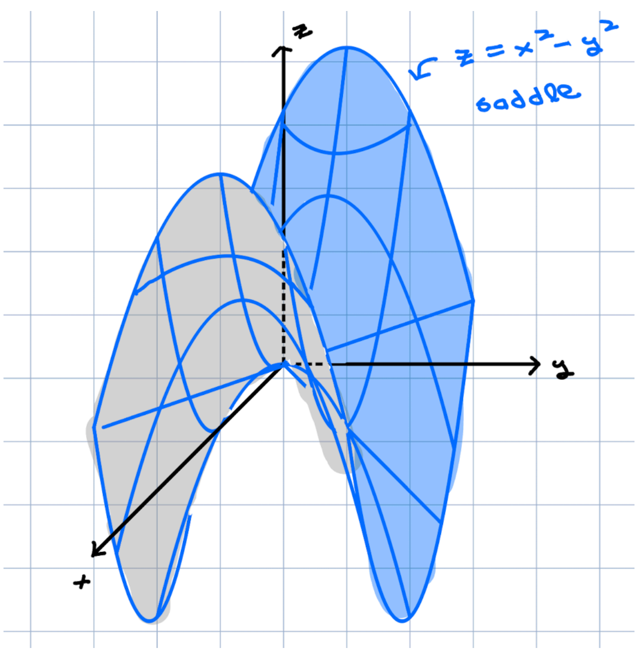


Making this sketch was time consuming. Very. Time-consuming.

C. **Saddles.** And now the equation $z = x^2 - y^2$ and its slices.



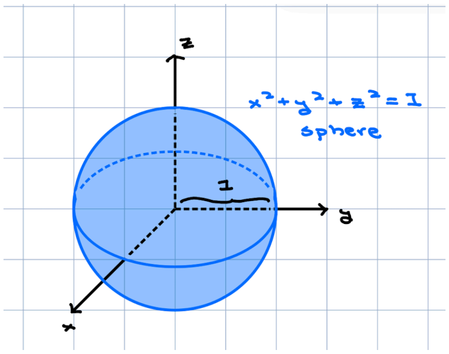
We put these together to obtain the surface it defines, which is technically called a **hyperboloic paraboloid** but is informally referred to as a **saddle**.



Can you imagine a little dude mounting this saddle on his little horse? Drawing this, again achieving some level of accuracy, was extraordinarily time-consuming. I wish I was an artist.

D. **Spheres and Hyperboloids.** The **unit sphere** is defined by the equation:

$$x^2 + y^2 + z^2 = 1$$

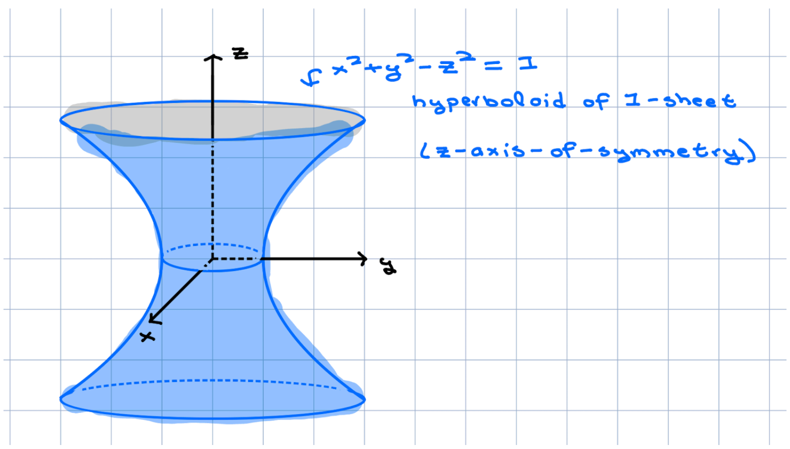


The equation $x^2 + y^2 + z^2 = 1$ is saying that the distance square of (x, y, z) from the origin is 1, which is why we obtain the unit sphere.

What if we change some of the signs in the lefthand side of the equation that defines the unit sphere? If 1 sign on the lefthand side of the sphere equation is flipped negative, as with:

$$x^2 + y^2 - z^2 = 1$$

we obtain the **hyperboloid of 1-sheet**:

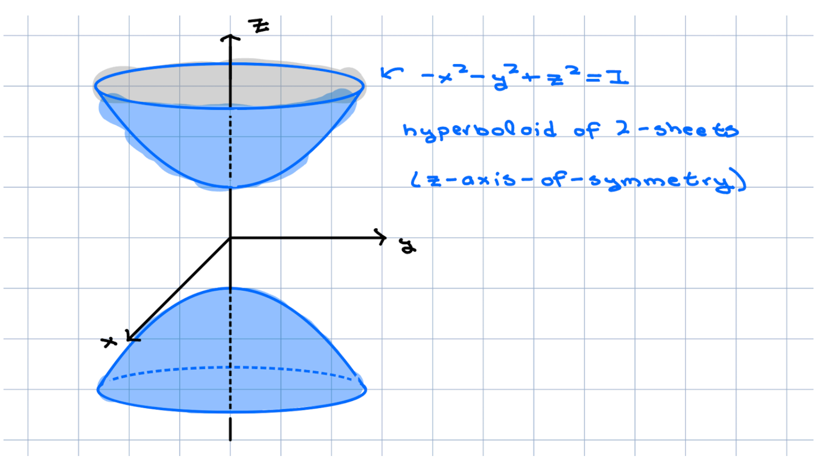


Its vertical slices are (usually) hyperbolas, and its horizontal slices are circles.

If 2 signs on the lefthand side of the sphere equation are flipped negative, as with:

$$-x^2 - y^2 + z^2 = 1$$

we obtain the **hyperboloid of 2-sheets**:



Its vertical slices are hyperbolas, and its horizontal slices are (usually) circles. But why does it not intersect the xy-plane? What happens if you set $z = 0$ in the equation? Compare this to setting $z = 0$ in the equation above of the hyperboloid of 1-sheet.

E. **Graphical Operations.** Surfaces can be stretched, shrunk, translated, and reflected. For example let us talk about the **ellipsoid** defined by:

$$\left(\frac{x-1}{2}\right)^2+\left(\frac{y-2}{3}\right)^2+\left(\frac{z-3}{4}\right)^2=1$$

in relation to the unit sphere $x^2+y^2+z^2=1$. Let's do it using desmos3D.

Desmos3D is a great resource if you would like to view a surface from multiple perspectives.

Here $\star$ refers to one of the variables: x, y, or z.

change made to $\star$ in equation	change made to surface
$\star \to \frac{\star}{a}$	scale by factor of a in the $\star$ direction
$\star \to \star - a$	translate by a in the $\star$ direction
$\star \leftrightarrow \star'$	reflect across plane $\star = \star'$

Note the general rule: the effect on the variable in the equation is the reverse of the effect on the surface. For example, replacing x with $\frac{x}{2}$ in the equation will actually stretch the surface by a factor of 2

We are now ready to discuss a classification of quadric surfaces.

Remember, a quadric surface is defined by an equation involving a two-variable polynomial of degree 2.

The names of the **non–degenerate** quadric surfaces are listed below. A surface has that name if it can be converted to the listed standard form through rotations and reflections.

Surface:	**Standard Form**	**Notes on Standard Form**
paraboloid:	$\frac{z-\ell}{c}=\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2$ compactly: $\hat{z}=\hat{x}^2+\hat{y}^2$	opens in pos. z–dirn
saddle:	$\frac{z-\ell}{c}=\left(\frac{x-h}{a}\right)^2-\left(\frac{y-\ell}{b}\right)^2$ compactly: $\hat{z}=\hat{x}^2-\hat{y}^2$	—
ellipsoid:	$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2+\left(\frac{z-\ell}{c}\right)^2=1$ compactly: $\hat{x}^2+\hat{y}^2+\hat{z}^2=1$	sphere if $a=b=c$
1–sheeted hyperboloid:	$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2-\left(\frac{z-\ell}{c}\right)^2=1$ compactly: $\hat{x}^2+\hat{y}^2-\hat{z}^2=1$	z–axis of symmetry 2 positive, 1 negative
2–sheeted hyperboloid:	$-\left(\frac{x-h}{a}\right)^2-\left(\frac{y-k}{b}\right)^2+\left(\frac{z-\ell}{c}\right)^2=1$ compactly: $-\hat{x}^2-\hat{y}^2+\hat{z}^2=1$	opens in z–directions 2 negative, 1 positive

You are not expected to know what **non-degenerate** means here. I am just using it to group an important list of surfaces. The cylinder from the very start is an example of a degenerate quadric surface.

You will see a few degenerate quadric surfaces in discussion, one of which is the **(elliptic) cone** which has standard form:

$$\left(\frac{z-\ell}{c}\right)^2=\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2$$

Example 1. Classify the surface defined by the equation by converting it to standard form, up to swapping of variables.

$4x^2 - 16y^2 + z^2 = -16$

This is a warning not to be too attached to merely counting the signs on one side of the equation without sparing a moment to think about the other side!

$(x + 3)^2 - y + z^2 = 1$