A. **Curves.** The 1D objects in space are called **curves**. Just like for straight lines, we will describe curves with **parametrizations**.

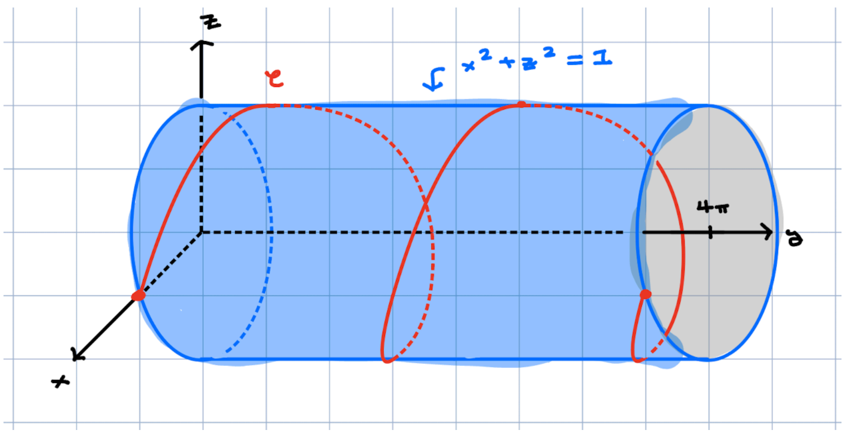
Remember: a parametrization is a function whose outputs are the position vectors of points along the curve.

Consider the curve $\mathcal{C}$ parametrized by:

$$\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle \quad \text{with} \quad 0 \leqslant t \leqslant 4\pi$$

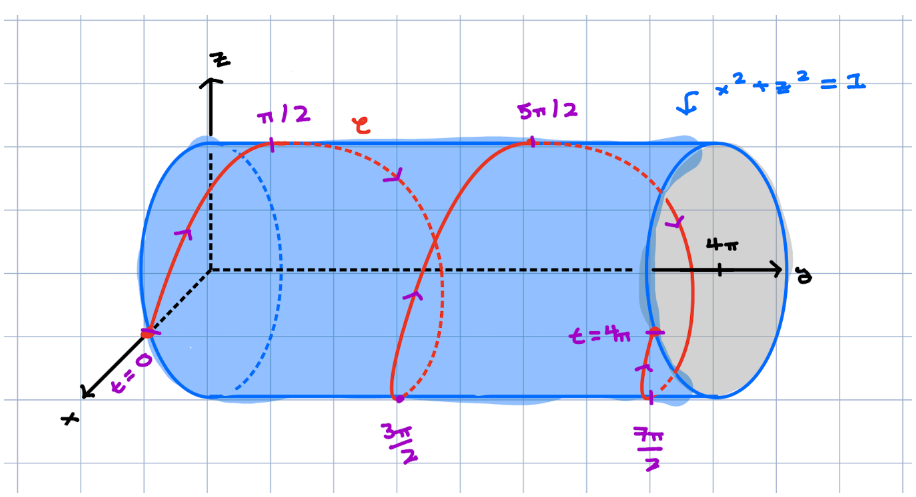
First let us verify that every point on this curve is on the cylinder $x^2 + z^2 = 1$.

Now we provide a sketch.



Note that the xz-coordinates circle around the z-axis, because they are $\cos t$ and $\sin t$ which literally trace out a circle!

The parametrization adds more detail then the physical curve itself.



The parametrization encodes an **orientation** (a choice of direction along the curve) as well as a **speed** (if you think of t as time. Both of these are not intrinsic to the physical curve in space, they are extra!

The parametrization $\mathbf{r}(t)$ is also called a **path** to distinguish it from fixed-in-space curve $\mathcal{C}$.

Example 1. Consider the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ with respective parametrizations:

$$\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$$

$$\mathbf{r}_2(t) = \langle 3-t, t-2, t^2 \rangle$$

Do $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect? And if so, at what point?

In order to check whether the curves intersect we need to check that any point at any parameter on the first curve does not match with any point at any parameter on the second curve. To set this up as an equation, we need to make sure we consider the possibility that the parameter on the first curve is different from the parameter on the second curve. This is why we change the name of one of the parameters from t to s.

Example 2. Parametrize the straight line segment from $P(2, 1, 3)$ to $Q(3, 4, 2)$.

The parametrization of a straight line segement from P to Q: $\mathbf{r}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ with $0 \leqslant t \leqslant 1$ is worth remembering.

B. **Path Derivatives.** Paths (parametrizations for curves) have speed and direction. Another word for this combination is **velocity**. This should ring a calculus bell: velocity is the derivative of position.

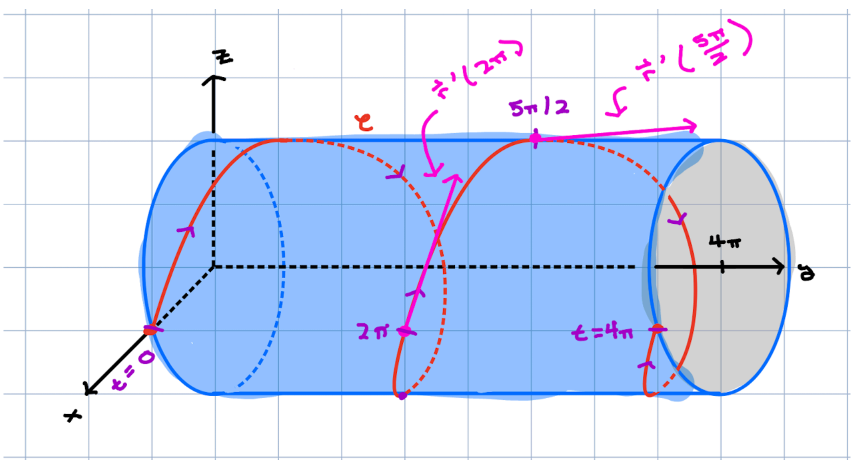
The **path derivative** is found by taking the derivative of each component of a path. For example if:

$$\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$$

then:

$$\mathbf{r}'(t) =$$

Let us look at a picture:



> When t is thought of as time and $\mathbf{r}(t)$ as position of a particle in space, then the path derivative $\mathbf{r}'(t)$ is also called the **velocity vector**. It points in the direction of the particle's path at time t and it length $\|\mathbf{r}'(t)\|$ is its **speed**.

Because $\mathbf{r}'(t)$ points in the direction of the particle's path along the curve, we also say that it is **tangent** to the curve.

Example 3. Let $\mathcal{C}$ be the curve parametrized by:

$$\mathbf{r}(t) = \langle \sqrt{t}, 2 - t, t^2 \rangle$$

Find a parametrization for the **tangent line** to the curve at the point:

$$P = (2, -2, 16)$$

Remember: the velocity vector $\mathbf{r}'(t)$ provides a tangent vector to the curve at the point on the curve at parameter t. So we will use that velocity vector as the direction vector for our line.

Example 4. In an earlier example you found that the paths:

$$\mathbf{r}_1(t_1) = \langle t_1, 1 - t_1, 3 + t_1^2 \rangle$$

$$\mathbf{r}_2(t_2) = \langle 3 - t_2, t_2 - 2, t_2^2 \rangle$$

intersect when:

$$t_1 = 1$$

$$t_2 = 2$$

Find the **angle** between the paths at this point of intersection.

By the angle between the paths, we mean the angle between their velocity vectors.