Example 1. Let $f(x, y, z) = y^2 \sin(x) + e^z$ and find:

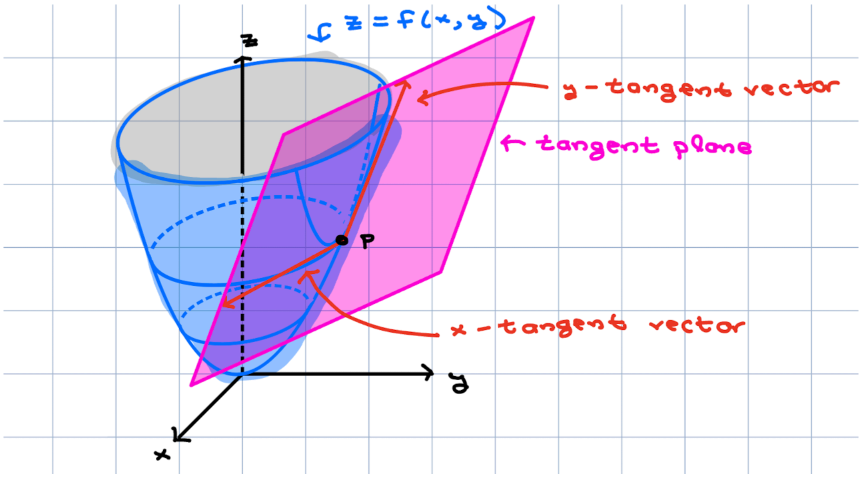
Even when there are more variables, to calculate $\frac{\partial f}{\partial \star}$ you should treat all other variables as constants and take a derivative with respect to $\star$.

$$\frac{\partial f}{\partial x} = f_x(x, y, z) =$$

$$\frac{\partial f}{\partial y} = f_y(x, y, z) =$$

$$\frac{\partial f}{\partial z} = f_z(x, y, z) =$$

A. **Tangent Planes.** The tangent vectors to $z = f(x, y)$ at (a, b) together form a plane called the **tangent plane**:



Remember we had calculated the x–tangent and y–tangent vectors to be:

x–tangent: $\langle 1, 0, f_x \rangle$

y–tangent: $\langle 0, 1, f_y \rangle$

and therefore a normal vector to the tangent plane is:

$\mathbf{n} = \Big[x\text{–tangent at } (a, b)\Big] \times \Big[y\text{–tangent at } (a, b)\Big] =$

The **tangent plane** to the $z = f(x, y)$ at $x = a$ and $y = b$ is defined by:

The equation for a plane with normal $\mathbf{n}$ through point P has equation:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

To concoct the equation for the tangent plane we used:

$\mathbf{n} = \langle -f_x(a, b), -f_y(a, b), 1 \rangle$

$P = (a, b, f(a, b))$

then computed the dot product, and moved things around a little.

When we zoom in to the point of tangency of the tangent plane, we see that the tangent plane is a good approximation for the surface, at least for x and y near $x = a$ and $y = b$.

The **linear approximation** for $f(x, y)$ that is suitable near $x = a$ and $y = b$ is:

$$L(x, y) =$$

We use the word **linear** because the tangent plane is defined by a linear equation. And this linear approximation is literally the same as the tangent plane, or more precisely, the graph of the linear approximation is the tangent plane. In other words, you find the linear approximation by solving for z in the equation of the tangent plane.

Example 2. Both parts relate to the function $f(x,y) = ye^{2xy}$.

(a) Find an equation for the tangent plane to $z = f(x,y)$ at the point $(0,1,1)$.

For this exact function we calculated $f_x(0,1) = 2$ and $f_y(0,1) = 1$ in an earlier example.

(b) Use part a and a linear approximation to find a rational number estimate for:

$0.9e^{0.18}$

A rational number is number that equals a fraction of integers. An integer is in the infinite list $0, \pm 1, \pm 2, \ldots$

B. **Higher-Order Derivatives.** There is nothing stopping us from taking repeated partial derivatives.

And if there is nothing stopping us, then why should we stop. Just keep at it until you draw your concluding breath on this small planet.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} =$$

$$f_x = \frac{\partial f}{\partial x} =$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} =$$

$f = xy^2 + e^y$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} =$$

$$f_y = \frac{\partial f}{\partial y} =$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} =$$

The **order** of a hihger-order partial derivative is the number of $\partial/\partial\star$'s it took to get you there. So $\partial^2 f/(\partial x \partial y)$ is a 2nd-order partial derivative.

Clairaut's Theorem: Where a function has continuous partial derivatives of all orders, the order in which we execute partial differentiation does not matter.

So for example:

$f_{yyx} =$

Virtually every function we encounter in this course will have continuous partial derivatives of all orders, at least where its partial derivatives of all orders are defined. So you can apply Clairaut's Theorem without further comment.

We have not even really talked about continuity of multivariable functions, so you are certainly not expected to verify it. You may just assume that we are working with nice-enough functions that, when you need Clairaut's Theorem, you may apply it. :)

Example 3. Find $f_{xyzzyxz}(x, y, z)$ where:

$$f(x, y, z) = \frac{z^2 e^{xy^2 + \sin(x+y)}}{x}$$

Only try a direct approach if you like the idea of self-inflicted pain.