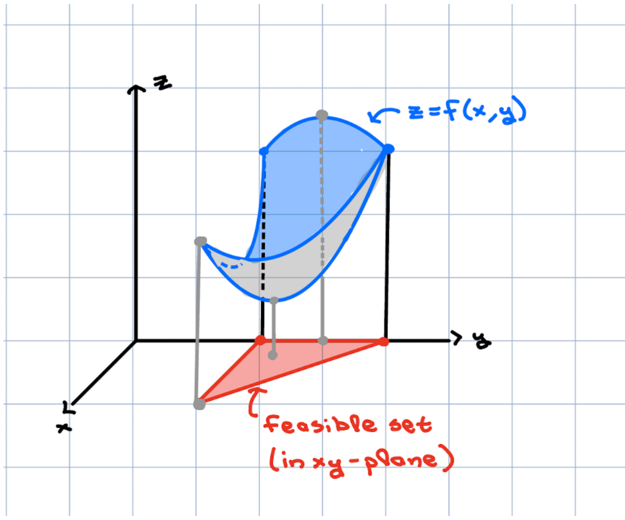
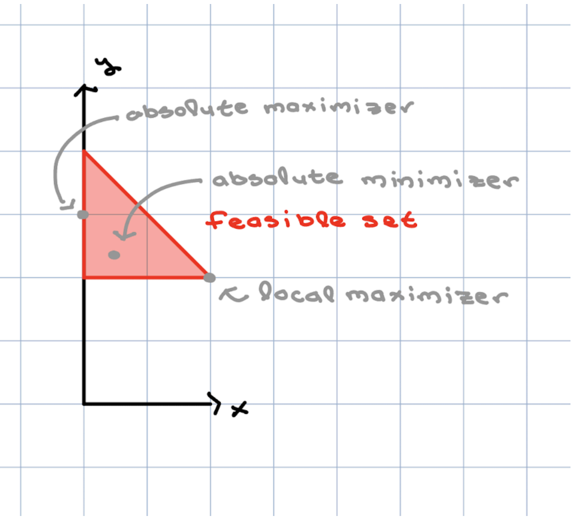
The word feasible comes from applications, where x and y might represent something like number of hamburgers and hotdogs sold, and it may only be feasible to sell between 0 and 12 hamburgers and hotdogs

A. **Global Extremums.** We would like to maximize and minimize a multivariable function $f(x, y)$ subject to the retriction that (x, y) belong to a given **feasible set**. We graph $z = f(x, y)$ for (x, y) in the feasible set.

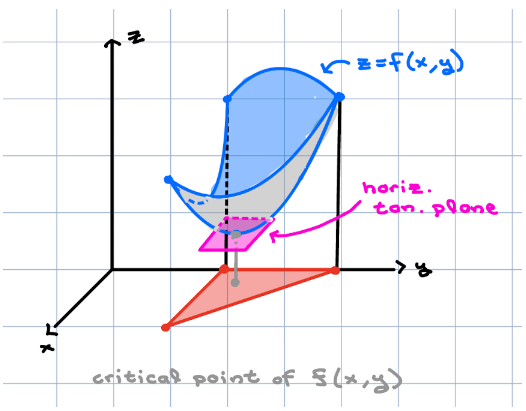


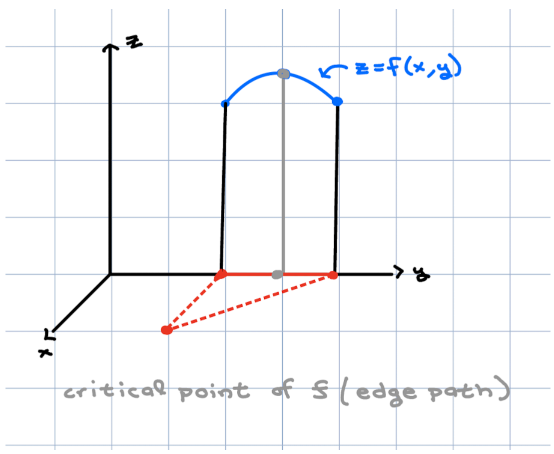


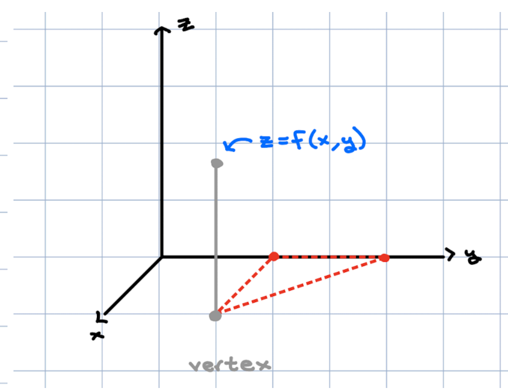
On the graph we have identified **absolute extremums** and **local extremums** and in the feasible set the corresponding **absolute extremizers** and **local extremizers**.

An extremum is either a maximum or minimum. An extremum is **absolute** if is most extreme among all points in the feasible set. An extremum is **local** if is is most extreme among all **nearby** points the feasible set. The **extremum** is the point on the graph, while the **extremizer** is the corresponding input from the feasible set.

Let us identify the different types of locations where we have extremizers.







A **critical point** of a differentiable multivariable function f is a point P where:

A critical point is a place where the tangent plane is horizontal, i.e. its normal vector is vertical. Earlier we found a normal vector to the tangent plane to $z = f(x, y)$ to be $\langle -f_x, -f_y, 1\rangle$. This vector will be vertical when $f_x = f_y = 0$, i.e. when $\nabla f = \mathbf{0}$.

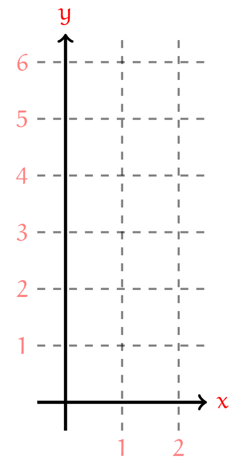
A differentiable function $f(x, y)$ on a feasible set only has extremizers at:

- critical points of $f(x, y)$ in the feasible set
- critical points of $f(\text{edge parametrizations})$
- vertices of the feasible set

Example 1. Consider $f(x,y) = x^2 - xy + y^2 - 3y$ on feasible set:

$$\mathcal{S} = \{(x,y) \mid 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 3x\}$$

The notation on the left is called **set notation**. It says that (x,y) is in the feasible set if and only if $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 3x$.



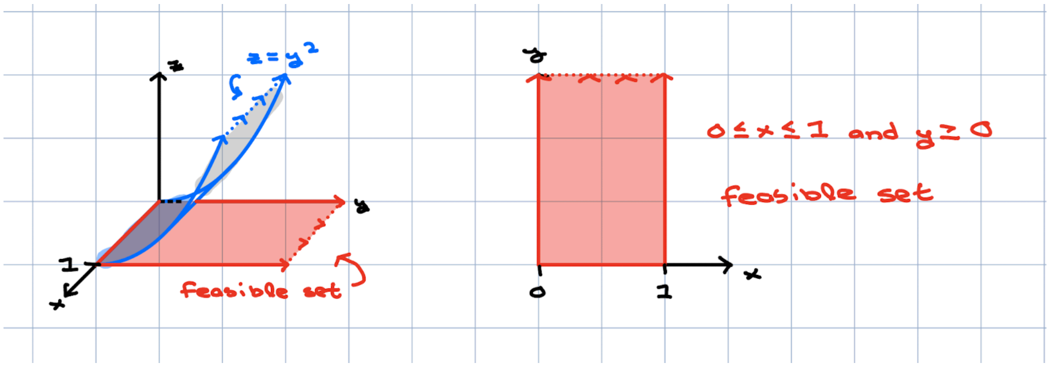
(a) Locate all critical points of $f(x,y)$ in the feasible set.

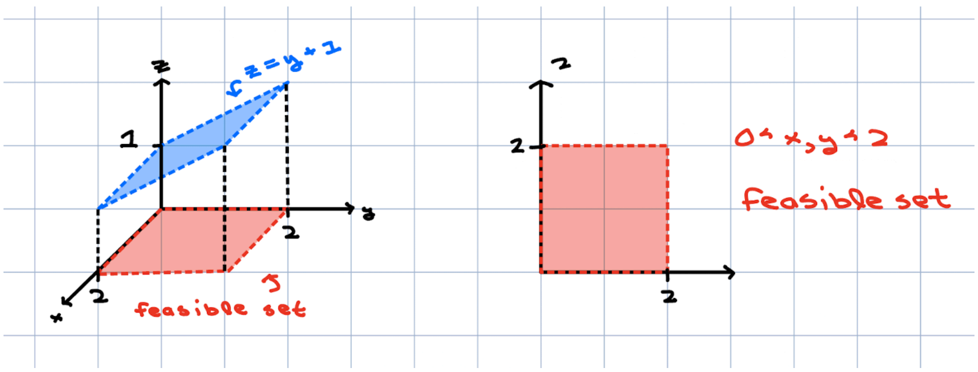
(b) Locate all critical points of $f(\text{edge parametrizations})$.

(c) Locate all vertices of the feasible set.

B. **Extreme Value Theorem.** Generally it is not guaranteed that a given function will achieve absolute extreme values on a given feasible set.

One reason we do not have this guarantee is because the function could fail to be continuous. In this course, however, virtually every function we encounter will be continuous where defined. We have not even discussed what it means for a multivariable function to be continuous. Suffice it to say, that functions built from continuous one-variable functions, but with extra variables and using simple algebraic operations, will be continuous where defined.





A straight dashed line indicates that the edge is **not** part of the set being considered

> A set is **bounded** if there is a finite upper limit to how far apart points in the set can be. Otherwise, it is **unbounded**.

> A point is on the **boundary** of a set if every disk of positive radius centered at that point, contains points both in the set and out of the set. A set is **closed** if every point that is on its boundary is also in the set.

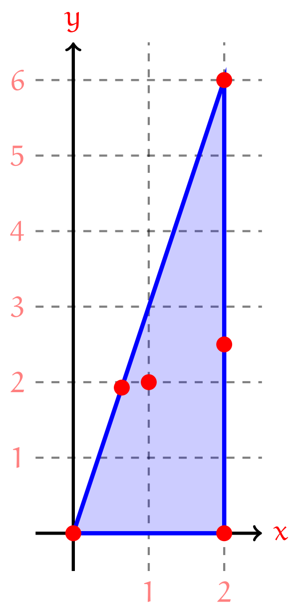
> **Extreme Value Theorem (   ).** A function that is continuous on a closed and bounded feasible set is guaranteed to achieve both absolute extremums on that set.

The terminology for a set that is both **closed** and **bounded** is **compact**. Hence the extreme value theorem says that a continuous function on a compact feasible set is guaranteed to achieve aboslute extremums.

Example 2. Earlier we had considered $f(x, y) = x^2 - xy + y^2 - 3y$ on feasible set:

$$\mathcal{S} = \{(x, y) \mid\ 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 3x\}$$

and gathered the following complete data:



point	type	f(point)
$(1, 2)$	critical point of $f(x, y)$	
$(\frac{9}{14}, \frac{27}{14})$	critical point of $f(\text{edge path})$	$-\frac{81}{28}$
$(2, \frac{5}{2})$	critical point of $f(\text{edge path})$	$-\frac{9}{4}$
$(0, 0)$	vertex	0
$(2, 0)$	vertex	4
$(2, 6)$	vertex	10

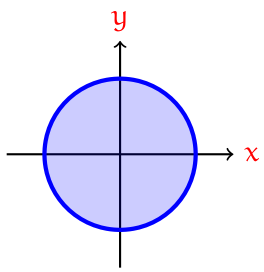
Well, I guess we did not compute the values in the last column, but all that needed to be done is to plug into $f(x, y)$!

Assess whether f achieves absolute maximums and minimums on $\mathcal{S}$, and find them if so.

Example 3. If they exist, find the maximum and minimum values of:

$$f(x, y) = xy$$

on the disk $D = \{x \mid x^2 + y^2 \leqslant 1\}$.



Remember that the unit circle is parametrized by $\langle \cos t, \sin t \rangle$ with $0 \leqslant t \leqslant 2\pi$.

In the solution we use the double-angle formula $\sin 2\theta = 2 \sin\theta \cos\theta$.