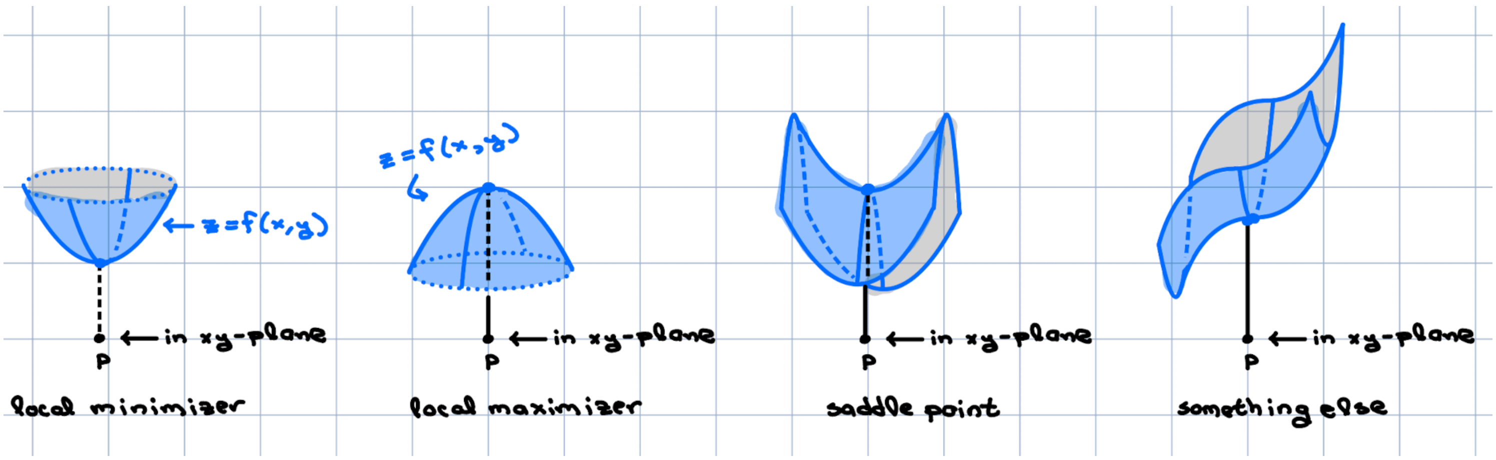
At a local minimizer: a function is concave up in all directions. At a local maximizer: a function is concave down in all directions. At a saddle point: a function is concave up in some directions, and concave down in other directions.

A. **2nd Derivative Test.** We now disengage focus from the restrictions imposed by a feasible set. A critical point of a differentiable function can fall into several important classes.



The second derivative is related to concavity and will help us with classifying critical points. The following is the relevant data from the second derivatives.

Concavity is about whether the graph bends upward like part of a $\smile$ (concave up) or bends downward like part of a $\frown$ (concave down).

The **Hessian** of $f(x, y)$ is the matrix:

$Hf(x, y) =$

Its **determinant** is: $\det Hf(x, y) =$

Its **trace** is: $\operatorname{tr} Hf(x, y) =$

2nd Derivative Test. If (a, b) is a critical point of $f(x, y)$ and:

- $\det Hf(a, b) > 0$ and $\operatorname{tr} Hf(a, b) > 0$ then (a, b) is a:
- $\det Hf(a, b) > 0$ and $\operatorname{tr} Hf(a, b) < 0$ then (a, b) is a:
- $\det Hf(a, b) < 0$ then (a, b) is a:
- $\det Hf(a, b) = 0$ then (a, b) could be:

If (a, b) is not a critical point, then it is not an extremizer or a saddle, so there is no point for our purposes to apply the second dervative test.

The idea behind the second derivative test is that when $\det Hf > 0$, concavity is the same in all directions, in which case you can determine the common direction of concavity (up or down) using the trace. And if $\det Hf < 0$ then concavity can be different in different directions, hence the saddle.

Technically for the second derivative test to apply, all of the second partial derivative need to be continuous. Again, in this course, this will virtually always be the case when we need to use the second derivative test.

Example 1. Find and classify all critical points of:

$$f(x, y) = 2x^3 + 3x^2 + 12xy + 3y^2 - 6y$$

Example 2. Find and classify all critical points of:

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$