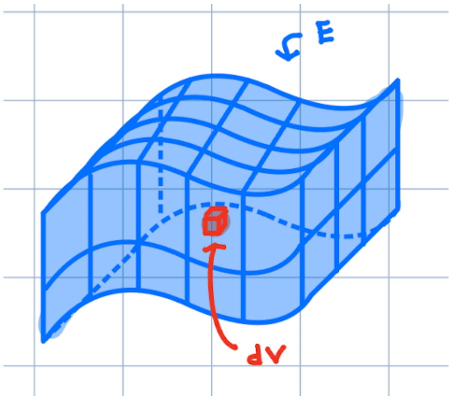
A. **Triple Integrals.** Let E be a 3D region and dV represent an infinitesimal bit of volume in that region.



If $f(x, y, z)$ represents density of [mass, electric–charge, ...] per unit volume at point (x, y, z) in the region E, then the **triple integral** of $f(x, y, z)$ over E is:

In particular:

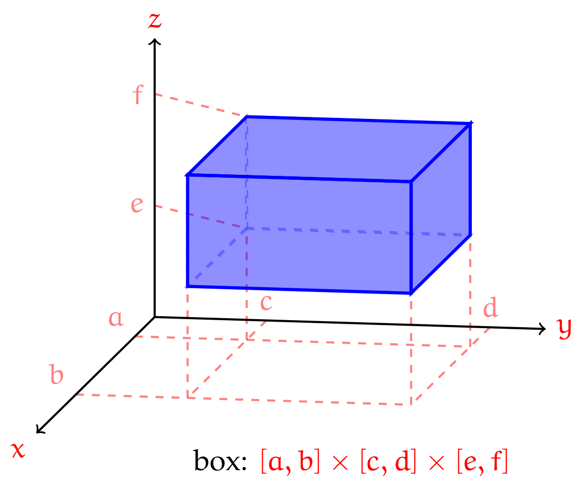
$$\iiint_E 1 \, dV =$$

The reason for three integral signs is that the region of integration is three–dimensional.

The units of $f(x, y, z)\ dV$ would then be [mass, electiric–charge, ...] specifically for that infinitesimal bit of volume. When all these infitenesimal quantities are added together using the integral, we obtain the total [mass, electiric–charge, ...] of the region.

If the density of mass per unit volume is 1, then the ratio of mass to volume is 1, meaning the mass of the region has the same value as the volume of the region.

B. **Triple Integrals Over Boxes.** Simple regions of **triple** integration are boxes.



box: $[a, b] \times [c, d] \times [e, f]$

If $E = [a, b] \times [c, d] \times [e, f]$ then:

$$\iiint_E f(x, y, z)\ dV =$$

The idea is that, if $f(x, y, z)$ represents density of mass per unit volume, we first compute the mass along a vertical segment in the region, then add the masses of these segments from left to right to obtain the mass of a vertical slice of the region, and then we add up the masses of all these vertical slices to obtain the total mass of the region.

You can change the order of integration and obtain the same answer, so long as the x-bounds line up with the dx-integral, the y-bounds line up with the dy-integral, and z-bounds line up with the dz-integral. Here are the other orders:

$$= \int_a^b \int_e^f \int_c^d f(x, y, z)\ dy\,dz\,dx$$

$$= \int_c^d \int_a^b \int_e^f f(x, y, z)\ dz\,dx\,dy$$

$$= \int_c^d \int_e^f \int_a^b f(x, y, z)\ dx\,dz\,dy$$

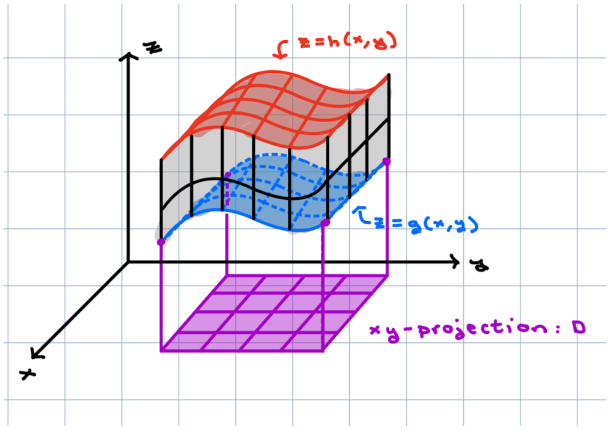
$$= \int_e^f \int_a^b \int_c^d f(x, y, z)\ dy\,dx\,dz$$

$$= \int_e^f \int_c^d \int_a^b f(x, y, z)\ dx\,dy\,dz$$

Example 1. If $E = [0, 1] \times [0, 2] \times [0, 3]$ then find:

$$\iiint_E 8xyz\ dV$$

C. **Triple Integrals Over Other Regions.** Just like with double integrals, it is not too much harder to deal with a region of integration that is between two graphs.



We call D the xy-projection, because it consists of all (x, y) so that (x, y, z) is in the region E. In this case, you can think of it as a shadow left by the region on the xy-plane, if there were a light emanating from far above.

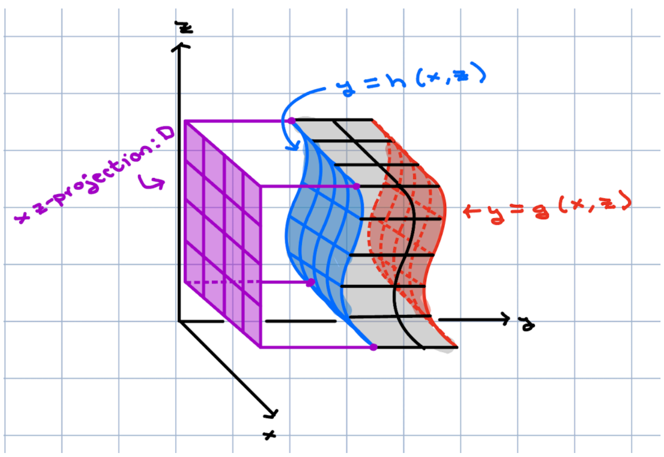
If E is consists of the vertical space above graph $z = g(x, y)$ and below graph $z = h(x, y)$ for all (x, y) in the region D, then:

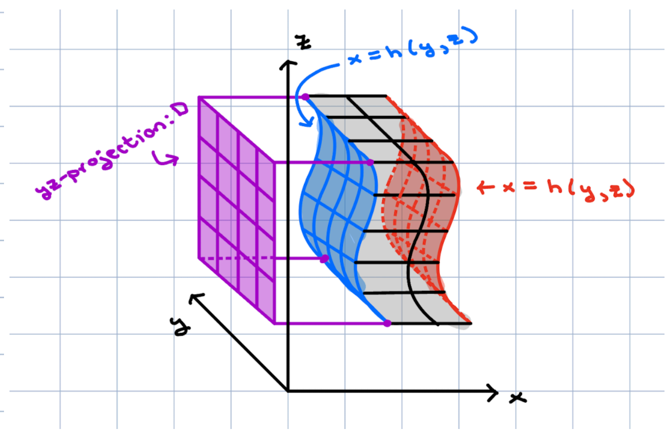
$$\iiint_E f(x, y, z)\, dV =$$

More precisely, this set E consists of all points (x, y, z) so that $g(x, y) \leqslant z \leqslant h(x, y)$ and (x, y) is in D.

The idea is that, if we think of $f(x, y, z)$ as density of mass, then the inner integral computes the mass along a vertical segment of the region at fixed x and y, and the outer integral adds up the masses of all of these vertical segments to obtain the total mass of the region.

We can also consider regions between graphs of functions of (y, z) or of (x, z).





If E is the set of (x, y, z) so that $h(x, z) \leqslant y \leqslant g(x, z)$ and (x, z) is in region D:

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{h(x,z)}^{g(x,z)} f(x, y, z)\, dy\, dA$$

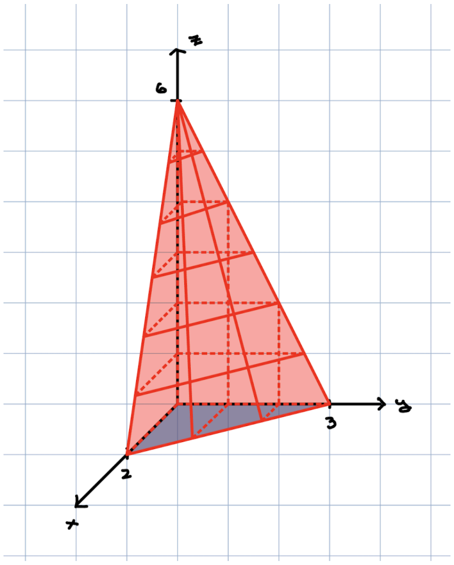
Here we call D the xz-projection, because it consists of all (x, z) so that (x, y, z) is in the region E. Here dA represents $dxdz$ or $dzdx$.

If E is the set of (x, y, z) so that $h(y, z) \leqslant x \leqslant g(y, z)$ and (y, z) is in region D:

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{h(y,z)}^{g(y,z)} f(x, y, z)\, dx\, dA$$

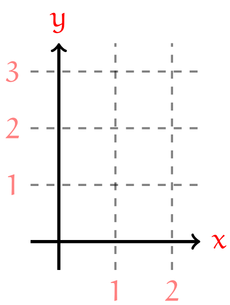
Here we call D the yz-projection, because it consists of all (y, z) so that (x, y, z) is in the region E. Here dA represents $dydz$ or $dzdy$.

Example 2. Use a triple integral to find the volume of the tetrahedron T bounded by the planes: $3x + 2y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.

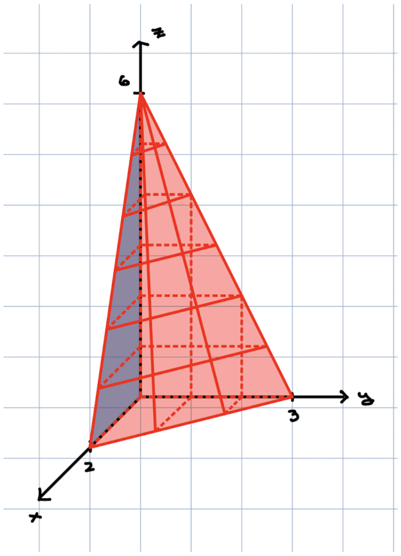


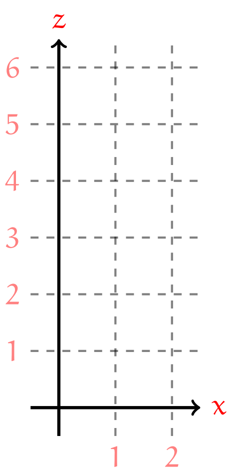
In this case the region is the vertical space between the bottom face and the slanted face, for all (x, y) in the region desribed by the bottom face.

In the order: $dz\,dy\,dx$



Instead, find the volume of the same tetrahedron T bounded by the planes: $3x + 2y + z = 6$, $x = 0$, $y = 0$, and $z = 0$, but in the order: $dy\,dz\,dx$.





Here we think of the tetrahedron as the left–right space, aka the y–space, between the left face and the slanted face.