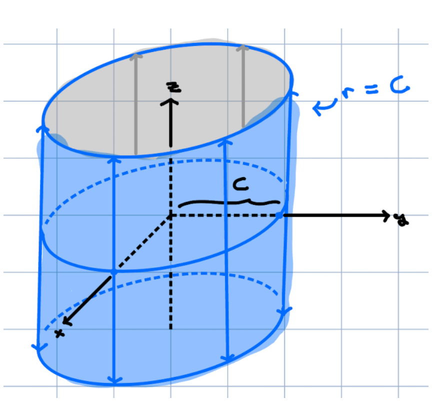
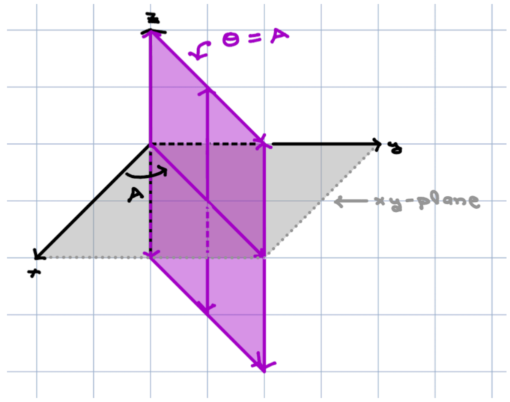
A. **Cylindrical Coordinates.** When we extend polar coordinates to three–dimensions by utilizing them as substitutes for xy–coordinates, but retain the z–coordinate, we get **cylindrical coordinates**.

Assuming r is nonnegative, the cylindrical coordinate level sets $r = C$ and $\theta = A$ look as follows.



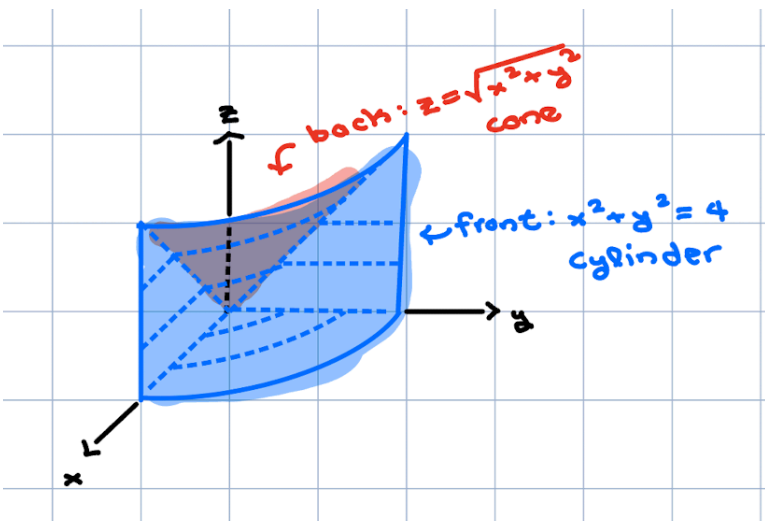


The level set $r = C$ is a vertical cylinder of radius C, centered on the z–axis. The level set $\theta = A$ is the vertical plane obtained by rotating the xz–plane by angle A counterclockwise when viewed from above.

Example 1. Let E be the region in the first octant under the cone $z = \sqrt{x^2 + y^2}$ and inside the cylinder $x^2 + y^2 = 4$.

The first octant is defined by $x, y, z \geqslant 0$.

Recall: $r^2 = x^2 + y^2$.

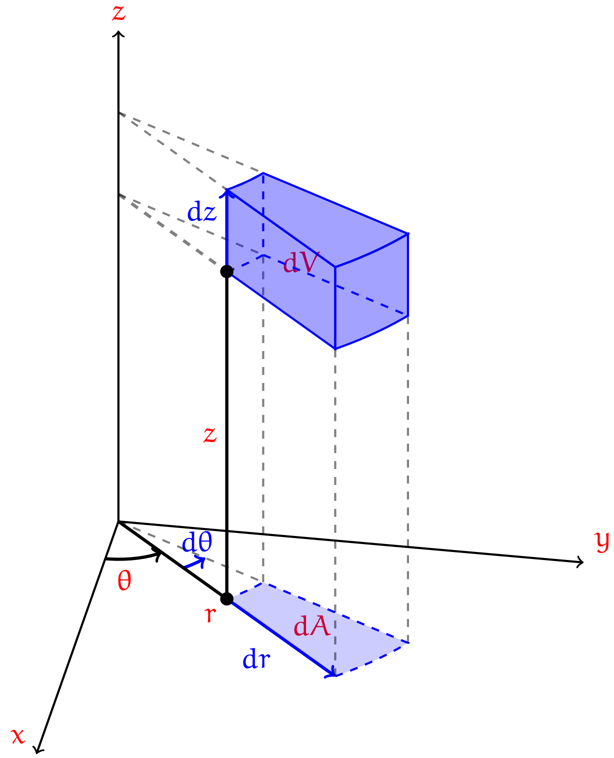


Describe inequalities in cylindrical coordinates that describe the region as if you were to compute an integral in order $dz dr d\theta$.

We have not talked about cylindrical integration yet, so let us just focus on the bounds.

When we set up bounds in this order, we need to express our region as being the vertical space above graph $z = g(r, \theta)$ and below graph $z = h(r, \theta)$ for all (r, θ) in a specific domain.

B. **Cylindrical Integration.** Now we will discuss integration in cylindrical coordinates. Imagine the infinitesimal bit of area dV obtained from the point with polar coordinates r, θ, z by increasing z by dz, θ by $d\theta$, and r by dr.



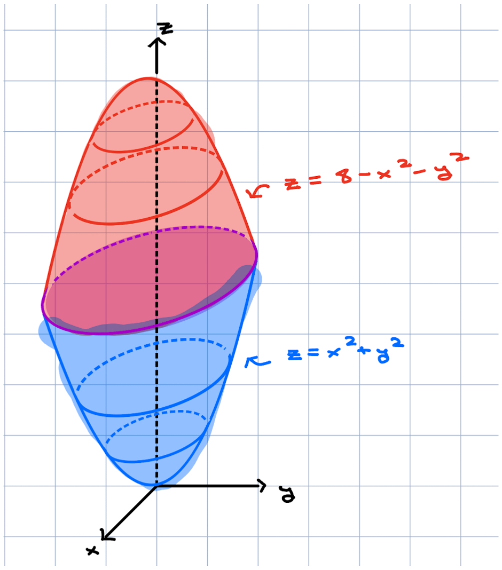
The goal in this process is to try to understand how small changes dr, $d\theta$, and dz are related to small changes dV in volume. This dV is exactly what appears in triple–integration.

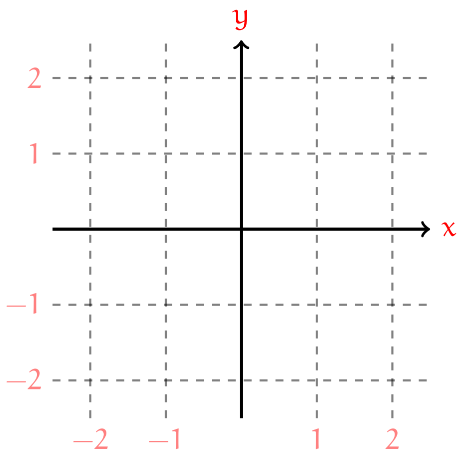
For example, we know that small changes dx, dy, dz effect the change $dV = dx dy dz$ in volume, essentially because the picture we would sketch in this case would be a rectangular box with sides dx, dy, dz.

In cylindrical coordinates:

$dV =$

Example 2. Let E be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

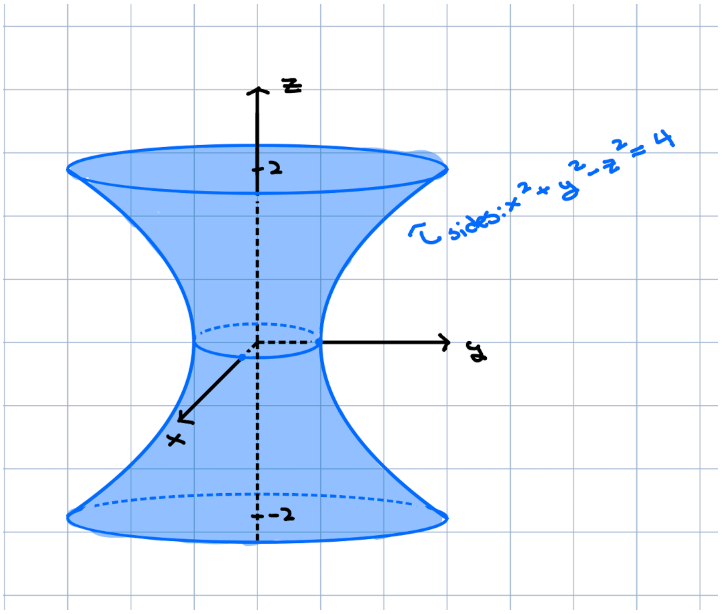




In this case our region is the vertical space between the bottom paraboloid and the top paraboloid, for all (x, y) inside the circle formed by their intersection.

Find: $\iiint_E z \, dV$

Example 3. Find the volume of region E bounded by the one–sheeted hyperboloid $x^2 + y^2 - z^2 = 4$ with $-2 \leqslant z \leqslant 2$.



Do not forget the you can obtain the volume by integrating $1\ dV$.

In this case it is not a good idea to set up the integral in the order $dzdrd\theta$ because this is **not** the vertical space between two graphs. You can see this issue by looking at the vertical gap in the left–right hyperbola.

Instead we realize that the hyperboloid is the graph of a function $r = h(\theta, z)$, and so try to set up the integral in order $drdzd\theta$.