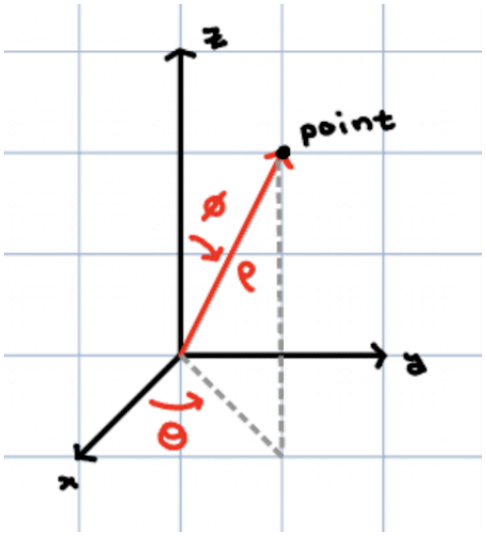
A. **Spherical Coordinates.** There is another coordinate system that is excellent for objects that are symmetric about the origin, like spheres centered at the origin. These are the spherical coordinates ρ, ϕ, and θ. The spherical coordinate θ is the same as the cylindrical coordinate.

ρ is read "roe" and ϕ is read "fee".



Because of the way spherical coordiantes are defined, if ρ is nonnegative, then it equals the distance of the point from the origin.

The point with spherical coordinates ρ, ϕ, and θ has position vector obtained by rotating the vector $\langle 0, 0, \rho \rangle$ down by angle ϕ in the vertical half–plane at angle θ. Every point can be described using coordinates in the range:

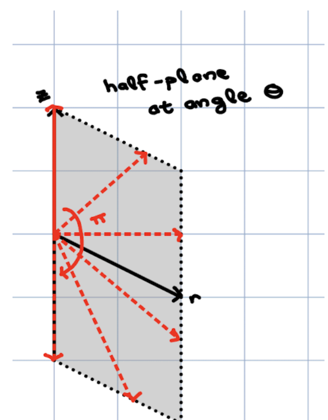
$\rho \geqslant$

$\leqslant \theta \leqslant$

$\leqslant \phi \leqslant$

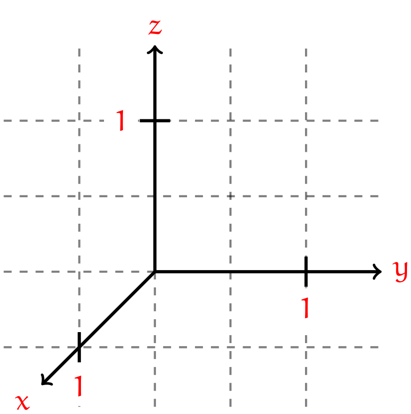
Specifically: the vertical half-plane is the level set of θ assuming that $r \geqslant 0$. We sketched these level sets when we first introduced cylindrical coordinate.

Every vertical half–plane can be described by θ in the range 0 to 2π. Every ray in that half–plane can be described by an angle ϕ down from the z–axis in range 0 to π.

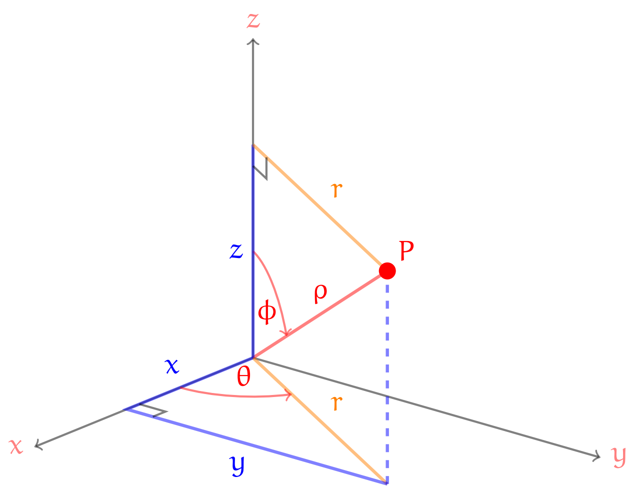


On a sphere, the coordinates ϕ and θ are analgous to latitude and longitude.

For example we sketch the surface $\mathcal{S}$ defined by $\rho = 1$, $0 \leqslant \phi \leqslant \frac{\pi}{2}$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$.



B. **Spherical Conversion.** We develop algebraic relationships between spherical coordinates and the other coordinate systems.



The two triangles formed by ϕ and by θ in this picture contain all the relations we describe below.

[Spherical $\leftrightarrow$ Cylindrical] and [Spherical $\leftrightarrow$ Standard]

r in terms of spherical coordinates:

x in terms of spherical coordinates:

y in terms of spherical coordinates:

z in terms of spherical coordinates:

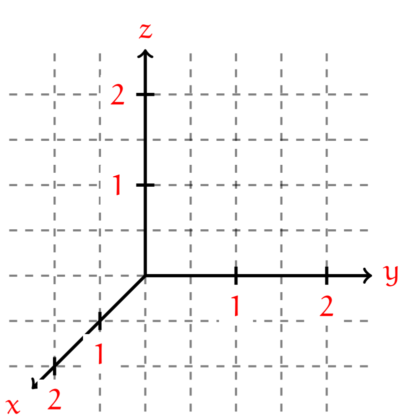
relation between ρ and standard/cylindrical coordinates:

relation between ϕ and standard/cylindrical coordiantes:

Example 1. Convert the cylindrical equation:

$$z = \sqrt{3}r$$

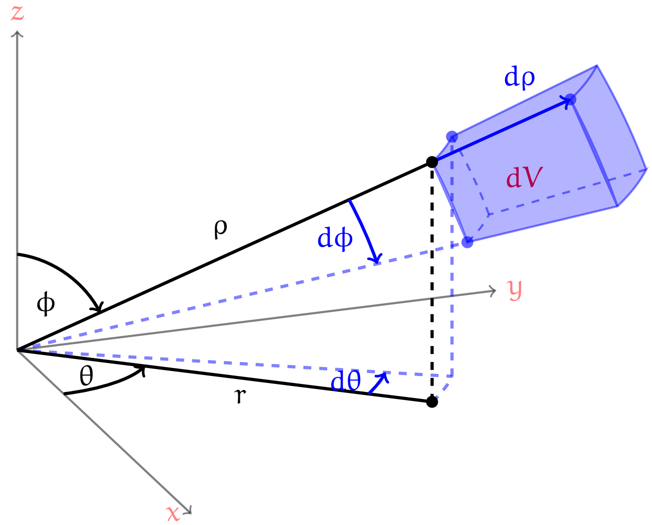
into a spherical equation, and sketch this surface assuming $r \geqslant 0$.



C. **Spherical Integration.** Now we will discuss integration in spherical coordinates. Imagine the infinitesimal bit of volume dV obtained from the point with spherical coordinates ρ, ϕ, θ by increasing ρ by $d\rho$, ϕ by $d\phi$, and θ by $d\theta$.

The goal in this process is to try to understand how small changes $d\rho$, $d\phi$, and $d\theta$ are related to small changes dV in volume. This dV is exactly what appears in triple–integration.

For example, we know that small changes dr, $d\theta$, dz effect the change $dV = r\, dr d\theta dz$ in volume.



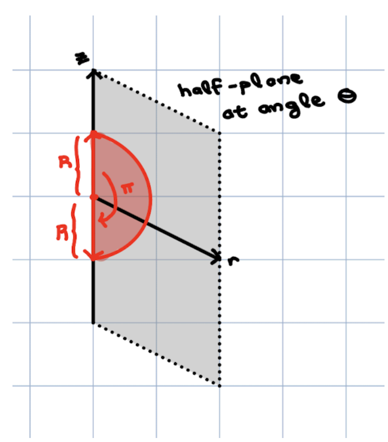
In spherical coordinates:

$dV =$

Example 2. Use spherical coordinates to derive the volume of a radius R sphere:

$$x^2 + y^2 + z^2 \leqslant R^2$$

To describe a solid sphere of radius R centered at the origin, the section of the sphere in each vertical half–plane at angle θ from 0 to 2π is traced out by rotating the line segment from $(0, 0, 0)$ to $(0, 0, R)$ down from angle 0 to angle π.



Example 3. Let E be the region outside the cone $z = \sqrt{x^2 + y^2}$ and inside the upper hemisphere of the sphere $x^2 + y^2 + z^2 \leqslant 1$. Find:

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}}\, dV$$

The upper hemisphere of this sphere, which is centered at the origin, consists of the half of the sphere above the xy-plane.