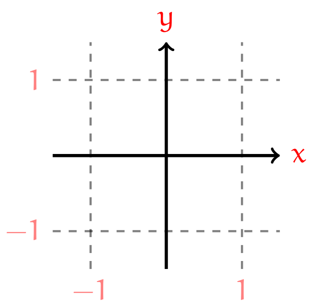
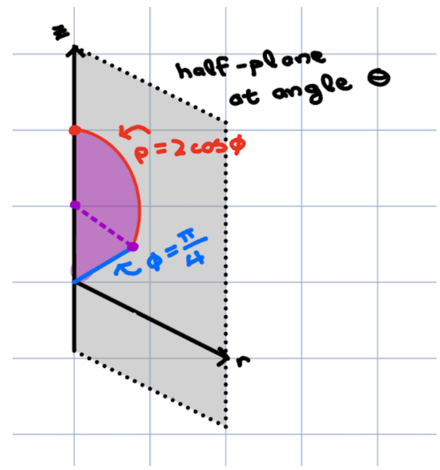
Example 1. Find the following by converting to spherical coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} 1 \; dz\,dy\,dx$$



The outer bounds describe the quarter disk $0 \leqslant r \leqslant 1$ with $0 \leqslant \theta \leqslant \pi/2$. For each θ in this range, the vertical half–plane at angle θ intersects the region as depicted below:



Recall the following relationships:

$\rho^2 = x^2 + y^2 + z^2$

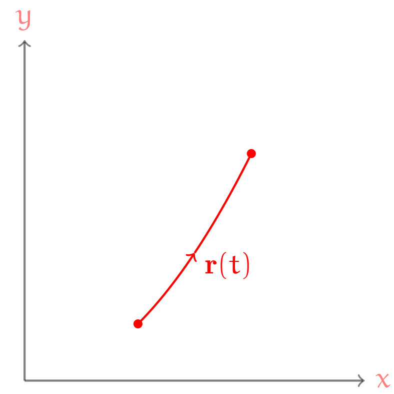
$\tan \phi = \dfrac{r}{z}$

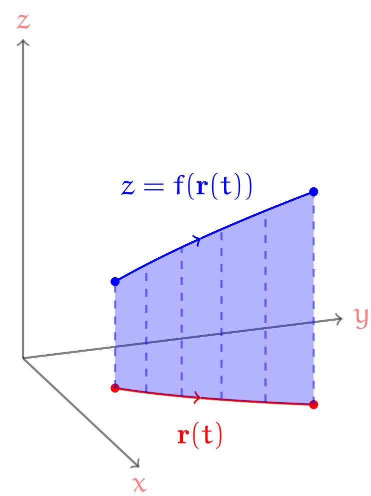
$dV = \rho^2 \sin \phi \; d\rho\, d\phi\, d\theta$

A. **Scalar Line Integrals.** We will transition next to focusing on integrating over curves: or more precisely over paths. We need a **path** $\mathbf{r}(t)$ with $a \leqslant t \leqslant b$ and a **function** $f(\mathbf{x})$ defined on the path outputs. With that we can construct a **fence** with base given by the path, and (signed) height given by the value of f along the path.

Remember that a path is parmetrization $\mathbf{r}(t)$ for a curve.

The input $\mathbf{x}$ could be (x, y) or (x, y, z) depending on whether the path is in two or three dimensions. Of course, when the path is in three dimensions, we cannot visually imagine the fence depicted, as it would technically require a fourth dimensions, however we will still be able to define the associated integral at the bottom of the page.





To find the fence area: first we find a small bit of **arclength** ds along our path.

Arclength is the word for length traversed along a path.

Remember that we could think of $\|\mathbf{r}'(t)\|$ as speed. And distance is speed times time.

$ds =$

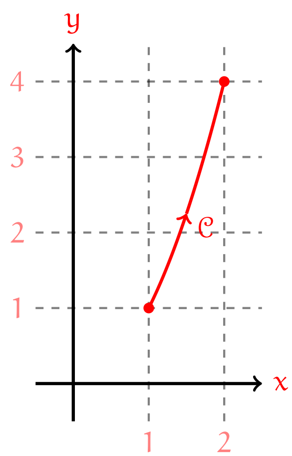
These infinitesimal bits of arclength form the base of our fence. To get area we multiply each of these by the height.

The **scalar line integral** of f along the path $\mathbf{r}(t)$ with $a \leqslant t \leqslant b$ is: $\int_{\mathbf{r}} f \, ds =$

This equals the signed area of the vertical fence between $f(\mathbf{r}(t))$ and the path $\mathbf{r}(t)$. Signed because when the fence dips below the curve, that area will be considered negative.

Example 2. Let $\mathcal{C}$ be the segment of the parabola $y = x^2$ from $x = 1$ to $x = 2$, and find:

$$\int_{\mathcal{C}} 8x \, ds$$



To find the line integral over a curve, we need to first select a parametrization for that curve. Because the fence area only depends on the curve, and not how we parametrize it, the scalar line integral will also only depend on the curve, and not how we parametrize it. So parametrize it in any way that floats your boat.

Example 3. Find the area of the fence above circle $x^2 + y^2 = 4$ in the xy–plane, and below the elliptic paraboloid $z = x^2 + 4y^2$.

The simplest way to parametrize a circle centered at the origin is to use the polar coordinate θ as our parameter.

To solve this we will use the power–reduction formulas:

$\cos^2\theta = (1 + \cos 2\theta)/2$

$\sin^2\theta = (1 - \cos 2\theta)/2$