A. **Fundamental Theorem of Line Integrals.** If you have a potential for a conservative vector field, then line integrals of that vector field are exceptionally easy to compute, as we will see below. Take a path:

$\mathbf{r}(t)$ with $a \leqslant t \leqslant b$

an a conservative vector field $\mathbf{F} = \nabla f$ be this conservative vector field. Then:

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} =$$

Our ingredients are the fundamental theorem of calculus, a classic:

$\int_a^b f'(t)\,dt = f(b) - f(a)$

and the multivariable chain rule, from your not–so–distant past:

$\frac{d}{dt}[f(\mathbf{r}(t))] = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$

Fundamental Theorem of Line Integrals. If $\mathbf{F}$ is conservative on the path $\mathbf{r}(t)$ with $a \leqslant t \leqslant b$, and has potential f, then:

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} =$$

Here end $= \mathbf{r}(b)$ and start $= \mathbf{r}(a)$.

Example 1. You found earlier that the vector field:

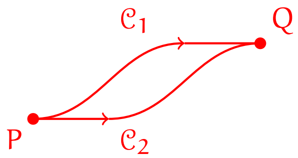
$\mathbf{F}(x, y, z) = (e^{yz} + z)\mathbf{i} + (xze^{yz} + 2yz)\mathbf{j} + (xye^{yz} + y^2 + x)\mathbf{k}$

has general potential function:

$f(x, y, z) = xe^{yz} + y^2z + xz + C$

Use this to compute $\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$ for the path $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ with $0 \leqslant t \leqslant 1$.

B. **Path Independence.** Suppose that $\mathbf{F} = \nabla f$ is conservative and we have two paths that have the same start and end.



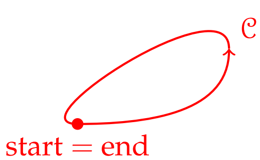
$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} =$$

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} =$$

> **Path Independence.** If $\mathbf{F}$ is **conservative** on path $\mathbf{r}$, then the line integral:
>
> $$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$$
>
> depends only on the start and end of the path, and nothing else.

In an earlier margin note we had stated that a conservative force is one such that the work done on a particle moving along a path only depends on the start and end of the path, not the path itself. Because work is the vector line integral of force, now we see where that comes from!

As a special case consider a **closed curve**: a curve whose start and end are equal.



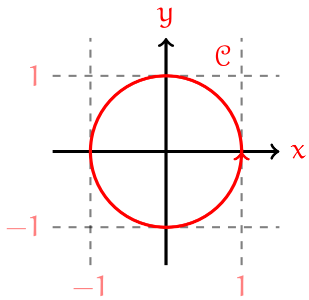
> **Closed Curves.** If $\mathbf{F}$ is conservative and $\mathcal{C}$ is **closed**, then:
>
> $$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} =$$

The $\oint$ symbol simply is an indicator that the curve being integrated over is closed, nothing more.

Example 2. Consider the vector field:

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

(a) Integrate $\mathbf{F}$ along the counterclockwise unit circle $\mathcal{C}$ centered at the origin.



This vector field is very interesting, because there are other indicators, which we will see shortly, that it could be conservative. In fact, it is conservative along any closed curve that does not circulate about the origin, and so will have vanishing integral over such a curve. The origin poses the problem.

(b) Is $\mathbf{F}$ a conservative vector field?

C. **Curl.** We introduce an object that measures the tendency of a vector field to circulate. We do this using **curl**. First we introduce the notation:

We call ∇ an **operator** because it exists to operates on functions, and does not have inherent meaning, or physicality, on its own. For example: ∇ operates on function f as ∇f, the gradient!

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \langle \partial_x, \partial_y, \partial_z \rangle$$

If $\mathbf{F}$ is a vector field, then its **curl**:

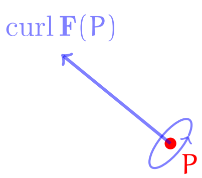
$$\operatorname{curl} \mathbf{F} =$$

The symbol $\times$ here is the cross product.

For example let's find $\operatorname{curl} \mathbf{F}$ where:

$$\mathbf{F}(x, y, z) = \langle xy \sin z, y - xe^z, xyz \rangle$$

What does curl measure? If $\mathbf{F}$ is a fluid velocity vector field, and P is a point, then there may be some tendency for fluid to **circulate** about P.



Without getting into specifics:

- $\operatorname{curl} \mathbf{F}(P)$ points in the direction determined by the righthand rule about which circulation is greatest
- the magnitude of $\operatorname{curl} \mathbf{F}(P)$ measures circulation per unit area

More precisely, if $\mathbf{u}$ is a unit vector rooted at P, then the dot product $\operatorname{curl} \mathbf{F}(P) \cdot \mathbf{u}$ measures the circulation about an infinitesimal disk orthogonal to $\mathbf{u}$ but divided by the area of that disk.