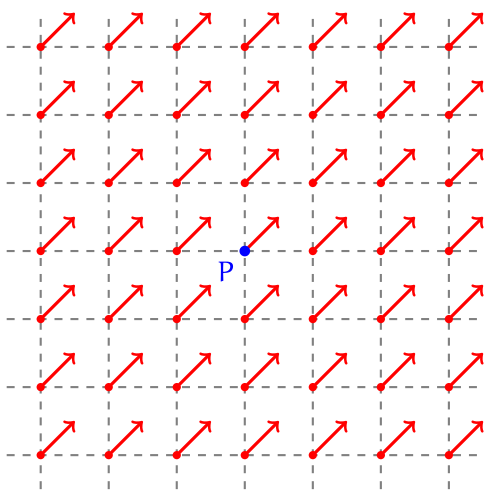
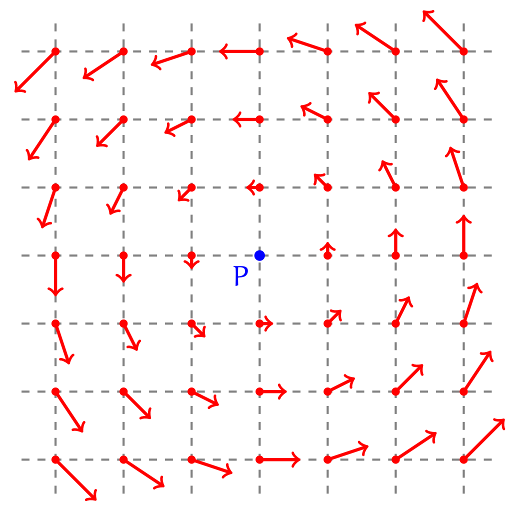
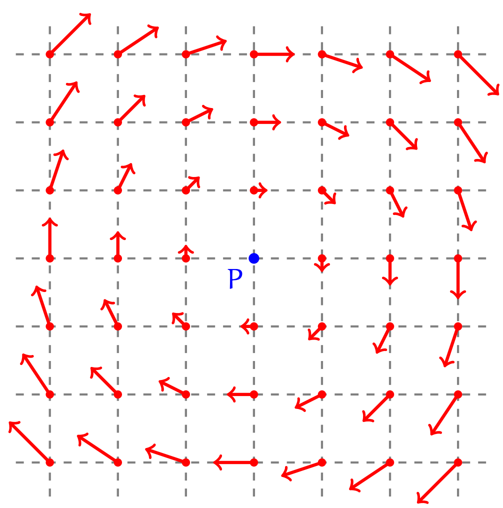
Example 1. For each vector field $\mathbf{F}$, decide whether $\operatorname{curl}\mathbf{F}(P)$ is $\mathbf{0}$, points into the page, or points out of the page.







A. **Curl and Conservative Vector Fields.** Let's calculate curl of a gradient:

$\text{curl}\,\nabla f(x, y, z) =$

Every conservative vector field is **irrotational**, meaning: $\text{curl}\,\mathbf{F} =$

Technically the vector field $\mathbf{F}$ should be continuously differentiable for this to hold. This is because it relies on the Clairaut's theorem, which concludes that the order of partial differentation does not matter, under certain simple assumptions.

We use this idea to show that the following vector field is **not** conservative:

$\mathbf{F}(x, y) = \langle -y, x \rangle$

For a two–dimensional vector field:

$\mathbf{F}(x, y) = \langle P, Q \rangle$

to compute the curl we treat the third component as 0, and thus:

$\text{curl}\ \mathbf{F}(x, y) = \text{curl}\,\langle P, Q \rangle = (Q_x - P_y)\mathbf{k}$

so that a two–dimensional vector field is irrotational if and only if:

$Q_x = P_y$

Example 2. Show that the following vector field is irrotational:

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

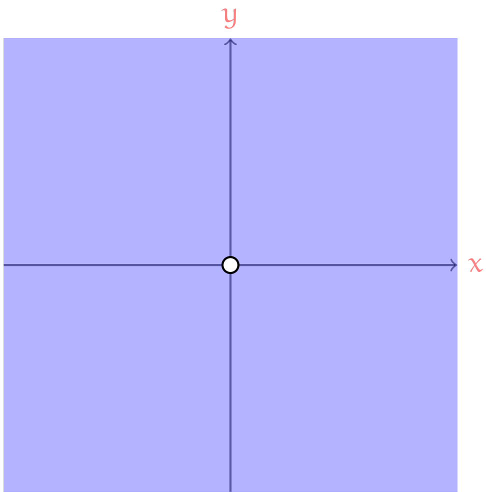
Recall the simplified formula for curl of a two–dimensional vector field:

$$\text{curl } \mathbf{F}(x, y) = \text{curl } \langle P, Q \rangle = (Q_x - P_y)\mathbf{k}$$

The vector field to the left was the interesting vector field we had explored before. We found it is **not** conservative, because it did not have path–independent line integrals, as its line integral over the counterclockwise unit circle was not zero, it was 2π! Remember that conservative vector fields are supposed to integrate to 0 over closed curves. Anyways. . . this vector field is the classic example of a vector field that is irrotational, but not conservative. Scary.

B. **Simple–Connectedness.** The previous example showcased how an irrotational vector field is not necessarily conservative. Preventing us from guaranteeing the conservativity is the type of region on which the vector field is defined.

The vector field $\mathbf{F}(x,y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$ is defined and irrotational on the region:



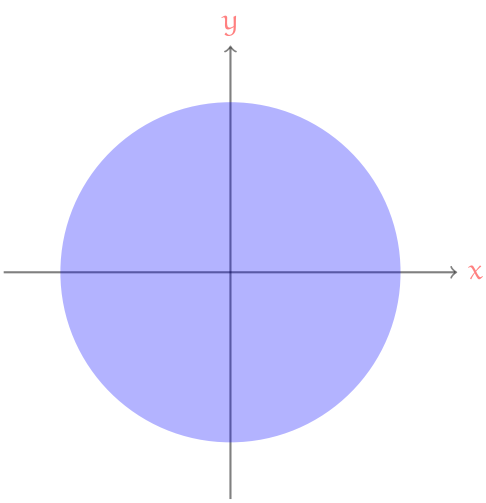
Path–independent line integrals for an **irrotational** vector field on a particular region is only guaranteed among paths that can be **continuously deformed** to one another while maintaining fixed endpoints and remaining with the region.

As a consequence, an irrotational vector field on a region is only guaranteed to have vanishing line integral on closed curve that can be **continuously deformed** to a point while remaining within the region.

A region is **simply–connected** if every closed curve in that region can be **continuosly deformed** within the region to a point in that region.

The idea behind continuous deformation is deforming the curve via bending, stretching, and shortening, but without cutting or gluing!

Is the region below simply–connected?



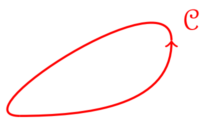
In a simplified but imprecise sense, a two–dimensional region is simply–connected if it contains no holes. This interpretation does not apply for three–dimensional regions.

Simply–Connected And Irrotational $\implies$ Conservative.
If $\mathbf{F}$ is defined and irrotational on a simply–connected region, then $\mathbf{F}$ is guaranteed to be conservative on that region.

The idea behind this result is that, in a simply–connected region, every closed curve can be continosly deformed to a point, and so by earlier margin notes, the irrotational vector field will have vanishing line integral over that curve. So the vector field will have vanishing line integral over every closed curve. This turns out to be enough to guarantee path–independence of line integrals, which is enough to guarantee the vector field is conservative.

Example 3. Let $\mathcal{C}$ be the curve in the xy–plane sketched below and find:

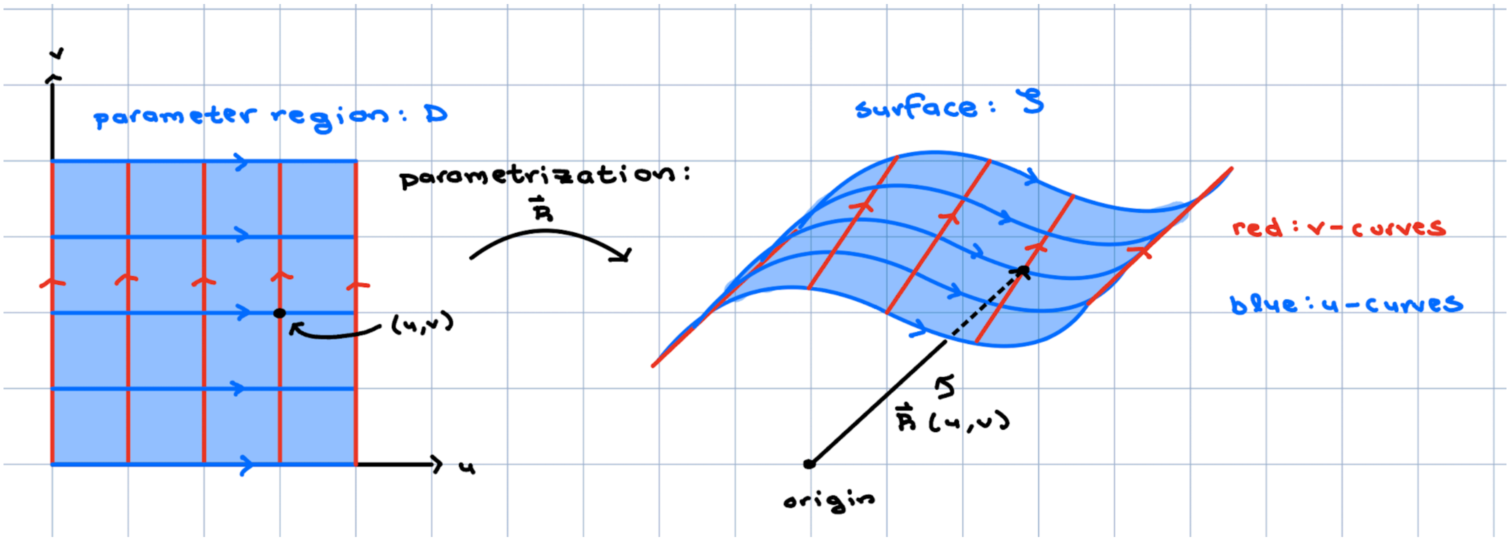
$$\oint_{\mathcal{C}} e^{\cos(e^x)}\, dx + \sin(\sin(e^{\sin y}))\, dy$$



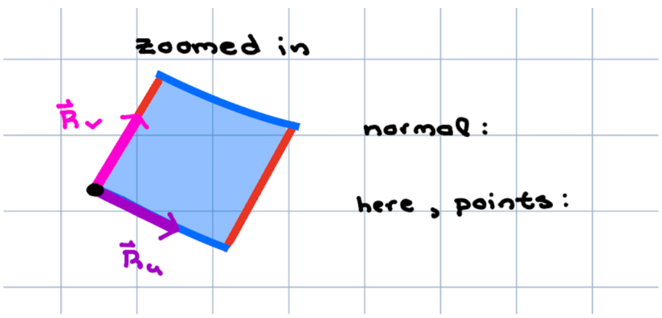
The set all points in the xy–plane is called $\mathbb{R}^2$ and is simply–connected. The set of all points in xyz–space is called $\mathbb{R}^3$ and is also simply–connected.

C. **Parametrized Surfaces.** We have engaged with curves for a while now. We now upgrade dimension and consider surfaces. We parametrize a surface $\mathcal{S}$ using a vector–valued function $\mathbf{R}(u, v)$ of two parameters u and v, where those parameters come from a specified parameter region D in the uv–plane.

One example of a surface is a plane, with which we have in fact played around.



Our next task is to locate tangent and normal vectors to the surface.

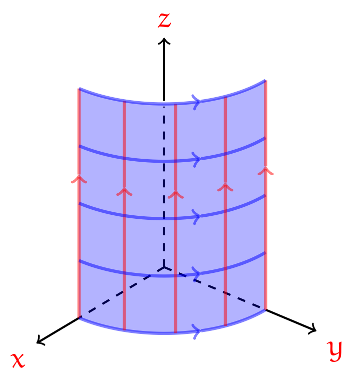


Here the partial derivatives $\mathbf{R}_u$ and $\mathbf{R}_v$ are tangent to the u–curves and v–curves respectively. To obtain a normal to the surface you would utilize a cross–product of tangent vectors.

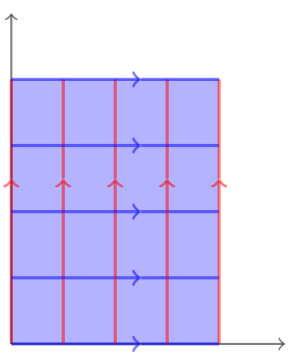
A choice of direction for normals to the surface is called an **orientation** for the surface.

Curves have an orientation, and so too do surfaces. In the picture of $\mathcal{S}$ at the top of the page, the cross product $\mathbf{R}_u \times \mathbf{R}_v$ determines upwards normals, which corresponds to an upwards orientation of the surface.

Example 4. Consider cylinder $\mathcal{S}$ given by $x^2 + y^2 = 1$ with $0 \leqslant z \leqslant 2$ and $x, y \geqslant 0$.



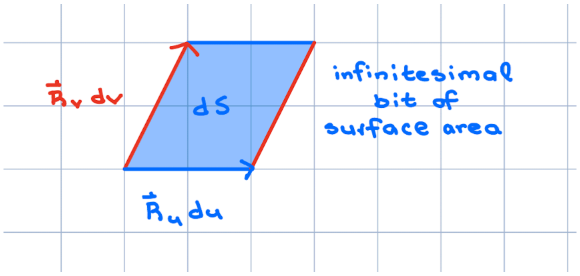
(a) Parametrize $\mathcal{S}$.



We will use an appropriate pair of cylindrical coordinates.

(b) Find a normal to $\mathcal{S}$ at the point with coordinates θ and z, and describe the corresponding orientation.

D. **Surface Area.** If we can parametrize a surface $\mathcal{S}$ with parametrization $\mathbf{R}(u,v)$ with a parameter region D, then its surface area is within reach.

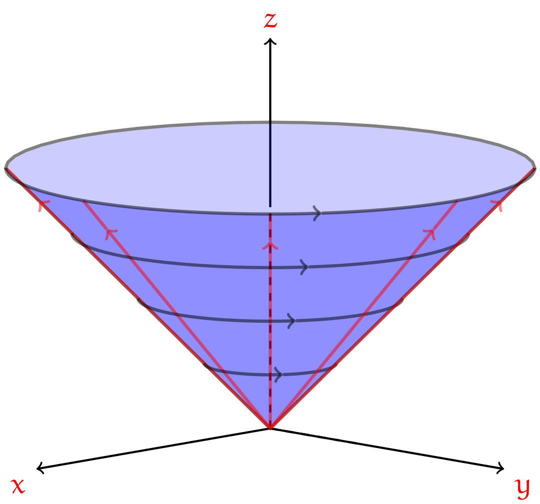


An infinitesimal of surface area can be described using a parallelogram, with sides given by an infitesimal bit of arclength along the u–curve and v–curve respectively. And of course, we cannot forget that one way to compute the area of a parallelogram formed by two vectors is to take the compute the length of their cross–product!

The surface area of a parametrized surface is:

Example 5. Find the surface area of the portion of the cone $z = \sqrt{x^2+y^2}$ with $0 \leqslant z \leqslant 2$.

We will use a pair of cylindrical coordinates to parametrize the cone.



Generally, if we parametrize graph $z = f(r)$ using parameters r and θ, then:

$$dS = r\sqrt{(f'(r))^2+1}\;dr\,d\theta$$