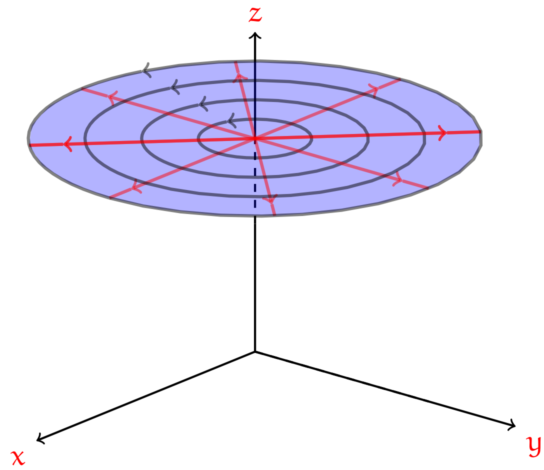
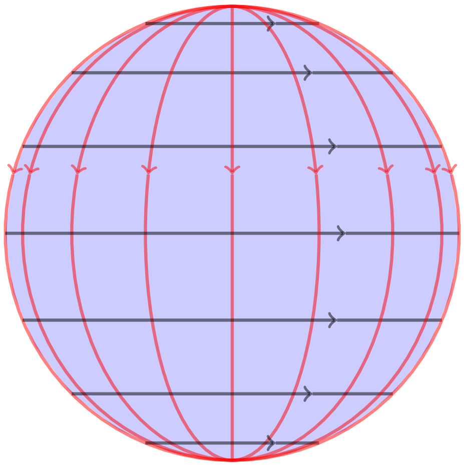
Example 1. $\mathcal{S}$ is surface $x^2 + y^2 \leqslant 1$ with $z = 1$ oriented with **upward** normals:



Let $\mathbf{F}(x, y, z) = e^{x^2+y^2+z^2}\mathbf{i} + \sin(x^2y^2)\mathbf{j} + 3\mathbf{k}$ and find: $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$

From an earlier margin note, we observed for a graph $z = f(x, y)$ parametrized with x and y that: $d\mathbf{S} = \pm\langle -f_x, -f_y, 1\rangle \, dxdy$

Example 2. Let $\mathcal{S}$ be the sphere $x^2 + y^2 + z^2 = 4$ oriented with **outward normals**:



Let $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ and find $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

Generally if the sphere of radius R centered at the origin is parametrized with ϕ and θ then:

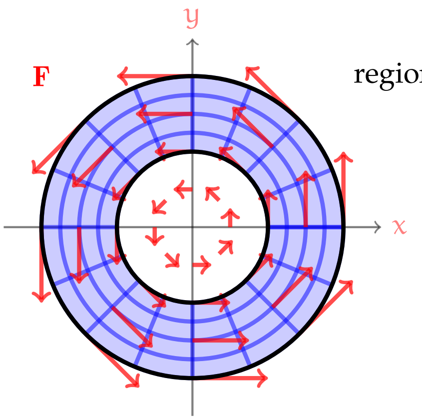
$$d\mathbf{S} = \pm R \sin\phi \, \langle x, y, z \rangle \, d\phi d\theta$$

$$dS = R^2 \sin\phi \, d\phi d\theta$$

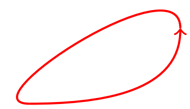
A. **Green's Theorem.** Our next task is to devise a relationship between line integrals and special kinds of double–integrals. Consider a region D in the xy-plane whose boundary ∂D is made up of **simple**, closed curves, all in the presence of a vector field $\mathbf{F}(x, y)$.

Recall that the boundary of a region in the xy–plane consists of thoses points so that, every disk centered at the region both contains points in the region, and points out of the region. The reason for the symbol ∂D to indicate the boundary is tied to a relation between boundaries and derivatives that will be partially revealed when we talk about Stokes's Theorem.

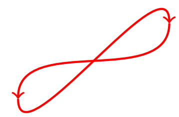
A closed curve is **simple** if it does **not** cross over itself.



region D with boundary ∂D

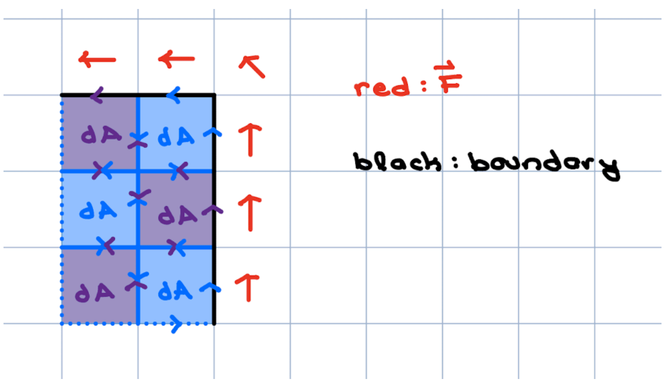


simple closed curve



closed curve, not simple

We divide our region into infinitesimal bits of area, and calculate the circulation about each infinitesimal bit.



Recall that curl has to to do with circulation. More precisely from an earlier margin note, if we compute the dot product $\text{curl}\,\mathbf{F} \cdot \mathbf{k}\, dA$ then we calculate the circulation about dA, using the righthand rule for out–of–page normals. Here $\mathbf{k} = \langle 0, 0, 1\rangle$ is the unit out–of–page vector.

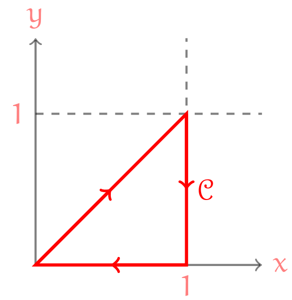
Also remember for a 2D vector field $\mathbf{F}(x, y) = \langle P, Q\rangle$ that $\text{curl}\,\mathbf{F} = (Q_x - P_y)\mathbf{k}$.

Green's Theorem. If D is a compact region in the xy–plane whose boundary ∂D is made up of simple, closed curves that have been oriented so that D is on the left, and if $\mathbf{F}(x, y) = \langle P, Q\rangle$ is defined and continuously differentiable throughout D, then:

The symbol $\oint$ here is an indicator that the curve is closed, nothing more.

Another way to think about the orientation is using the righthand rule with out–of–page normals: situate your righthand along the curve so that your thumb is pointing out–of–the page. If the region is on the left of your hand, then the your hand points in the correct orientation. If the region is on the right of your hand, then your hand points opposite the correct orientation.

Example 3. Let $\mathcal{C}$ be the closed curve depicted below, with clockwise orientation.



Use Green's Theorem to find: $\oint_{\mathcal{C}} y(1-e^{x^2})\ dx + x\ dy$

Before we dive in, we make the observation that this is a simple closed curve, so is suitable for Green's Theorem. If you forget, simple means the curve is not self–intersecting. In fact the notation $\oint$ indicates that the curve we are integrating over is closed.

Next we find a region D having $\mathcal{C}$ as its boundary. And then we assess whether the curve has a compatible orientation with Green's Theorem.