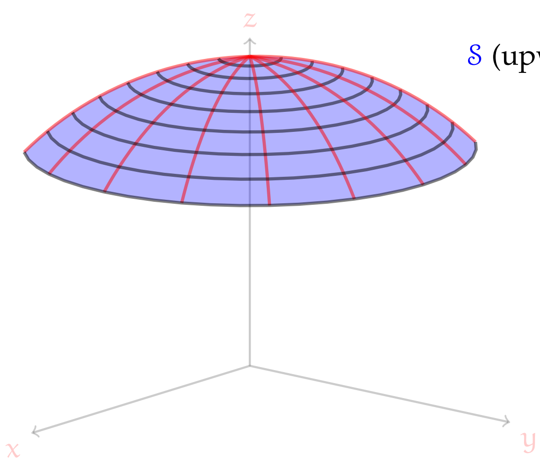
Example 1. Let's consider the exact same problem once again:

$$\mathbf{F} = -y\mathbf{i} + \left(x + (z-1)xe^x\right)\mathbf{j} + \left(x^2 + y^2\right)\mathbf{k}$$

and $\mathcal{S}$ be the spherical cap $x^2 + y^2 + z^2 = 2$ with $z \geqslant 1$ and oriented with upward normals.

Recall the shortcut formula for a surface $z = f(x, y)$ parametrized by x and y:

$\mathbf{dS} = \pm\langle -f_x, -f_y, 1\rangle\, dx dy$



$\mathcal{S}$ (upward normals)

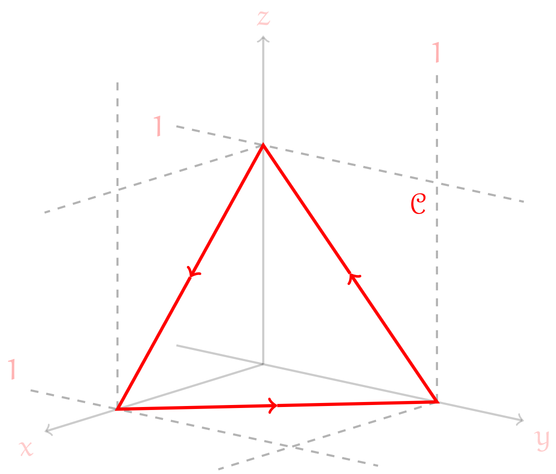
Curls Have Surface–Independent Integrals. If $\mathcal{S}$ and $\mathcal{S}'$ are oriented surfaces with a common boundary, so that the surfaces are on the same side of the boundary when viewed so their normals are pointing at you, then:

The idea is that for curl vector fields, and only for curl vector fields, we can compute a surface integral into an integral over the boundary. In other words, the surface itself does not matter, only its boundary!

Let's use this to solve the same problem another way. Pick a better surface with the same boundary to find:

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} =$$

Example 2. Let $\mathcal{C}$ be the oriented curve of straight line segments $(1,0,0) \to (0,1,0) \to (0,0,1) \to (1,0,0)$.



Use Stokes's Theorem to find: $\oint_{\mathcal{C}} \left(x\sin(e^x) - xz\right)\, dx - xy\, dy + (z^2 + y)\, dz$.

Remember that the symbol $\oint$ just indicates the curve is closed, nothing more.

In this case we find a surface $\mathcal{S}$ having $\mathcal{C}$ as its boundary, and select an orientation for that surface so that $\mathcal{S}$ is on the left of $\mathcal{C}$, when viewed from a perspective so the normals are pointing at you. Then Stokes's Theorem let us write:

$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$

In this case, the curl of $\mathbf{F}$ will be significantly nicer than $\mathbf{F}$ itself, which is why the Stokes's Theorem approach is recommended.

Do not forget that if we parametrize surface $z = f(x,y)$ with x and y then:

$$d\mathbf{S} = \pm\langle -f_x, -f_y, 1\rangle\, dxdy$$