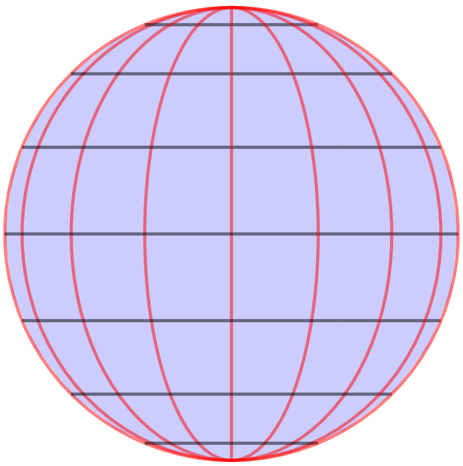
Example 1. Earlier we said that a surface is **closed** if it has an interior that was impossible to enter. Now we can be more precise.

> A surface is **closed** if it bounded and has **no** boundary curves.

For example the full sphere $\mathcal{S}$ depicted below is a closed surface.



Over a compact, closed surface, the circulations $\operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ about each infinitesimal bit of surface area will **all** cancel out, because there is no boundary.

> If $\mathbf{F}$ is a vector field that is defined and continuously differentiable throughout a **closed** surface $\mathcal{S}$, then
>
> $\oiint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} =$

The $\oiint$ symbol is just an indicator that the surface being integrated over is closed, nothing more.

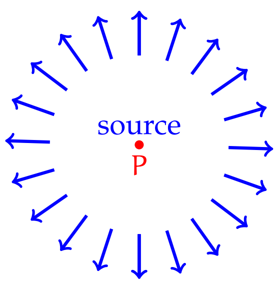
A. **Divergence.** When $\mathbf{F}$ represents fluid flow we may wish to capture the extent to which fluid is flowing away from a given point. To find this we use **divergence**.

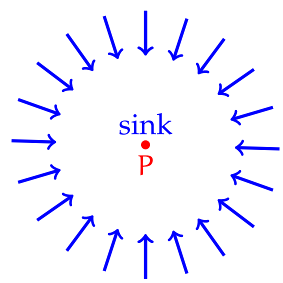
$\operatorname{div} \mathbf{F} =$

For example if $\mathbf{F} = (xy \sin z + y)\mathbf{i} + (y - xz^2)\mathbf{j} + (xyz)\mathbf{k}$ then:

$\operatorname{div} \mathbf{F} =$

If $\mathbf{F}$ measures fluid flow, and P is a point, then there may be some tendency for fluid to **flow away from or towards** P.



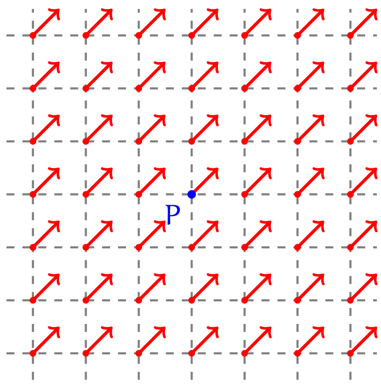


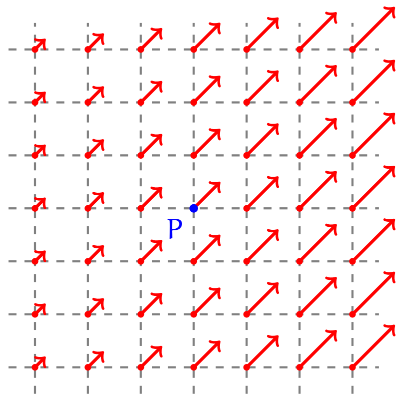
Without getting into specifics $\operatorname{div} \mathbf{F}(P)$ measures the:

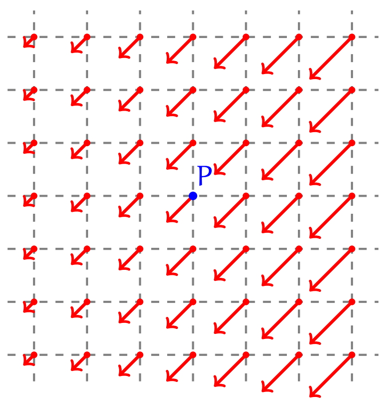
- **net** flow rate from P
- flow away = positive
- flow towards = negative

More precisely, if $\mathbf{F}(x, y)$ is two–dimensional, then $\operatorname{div} \mathbf{F}(P)$ measures the rate of flow out of an infinitisemal disk centered at P, divided by the area of that infinitesimal disk. If $\mathbf{F}(x, y)$ is three–dimensional, then $\operatorname{div} \mathbf{F}(P)$ measures the rate of flow out of an infinitisemal ball centered at P, divided by the volume of that infinitesimal ball.

Example 2. For each $\mathbf{F}$, decide whether $\operatorname{div}\mathbf{F}(P)$ is zero, positive, or negative.







Example 3. Let $\mathbf{F}$ be a twice–continuously differentiable vector field and find:

$\operatorname{div}\operatorname{curl}\mathbf{F} =$

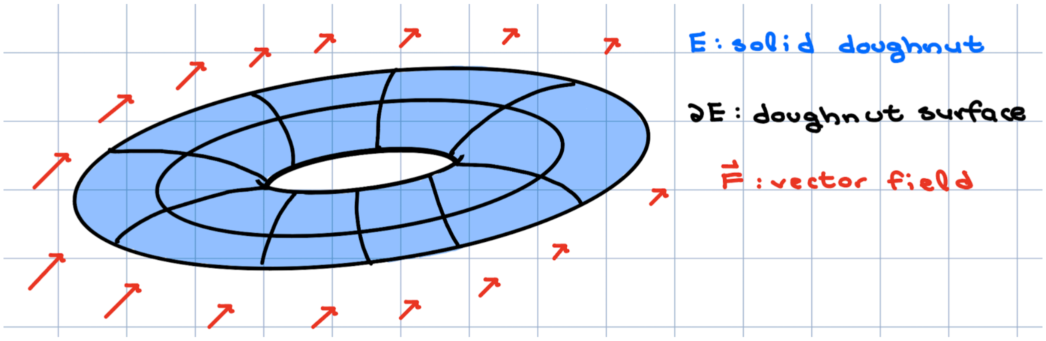
So the divergence of a curl vector field is always 0. This is analagous to how the curl of a gradient vector field (i.e. a conservative vector field) is always $\mathbf{0}$, as we had learned back when we worked with conservative vector fields all those years ago.

The intuition for why this is true is, the curl points along the axis of circulation, and does not change significantly along an infinitesimal bit of this axis. Therefore, near any point, the curl vector field roughly points in the same direction, and the vectors have roughly the same length. So, near that point, the image is like the first vector field from the last page. This results in divergence zero.

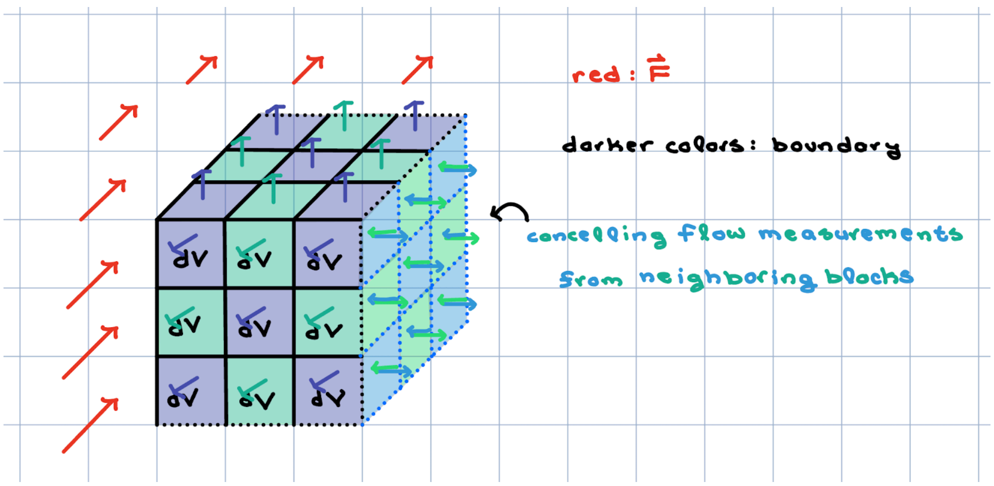
B. **Divergence Theorem.** Let E be a compact solid 3D region whose boundary is a closed surface ∂E, all in the presence of a vector field $\mathbf{F}$.

A point is on the boundary of a 3D region if every ball centered at the point enters the interior and exterior of the region.

Doughnuts are delicious but potentially harmful to health. Consume with care.



We divide the solid region into infinitesimal bits of volume dV, and calculate the flow out of each:



Based on an earlier margin note, the quantity $\operatorname{div}\mathbf{F}\,dV$ measures the flow out of the block dV. Or, more specifically, it calculates the net flow across the surface of the block, with flow out of the block measured as positive, and flow in measured as negative.

Divergence Theorem: If ∂E is oriented with **outward** normals and $\mathbf{F}$ is a vector field that is defined and continuously differentiable throughout the solid E, then:

The idea is that when we add all the $\operatorname{div}\mathbf{F}\,dV$ using a triple–integral, then, due to internal cancellation thanks to neighboring bits of volume, only the flow across the boundary of the surface (with out measured as positive) will survive.

Example 4. If $\mathcal{S}$ is the surface $x^2 + y^2 + z^2 = 4$ with **outward** normals and:

$$\mathbf{F} = \left(x^3 + ye^{\sin\left(yz^2\right)}\right)\mathbf{i} + \left(y^3 + xe^{\sin\left(xz^2\right)}\right)\mathbf{j} + \left(z^3 + ye^{\sin\left(yx^2\right)}\right)\mathbf{k}$$

Then find $\oiint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

Remember that if you want to compute a triple integral using spherical coordinates, then $dV = \rho^2 \sin\phi \; d\rho d\phi d\theta$.

Also, the ball of radius R centered at the origin is described by inequalities $0 \leqslant \rho \leqslant R$ and $0 \leqslant \phi \leqslant \pi$ and $0 \leqslant \theta \leqslant 2\pi$.