Math 226 Final Exam

Fa24

Wed Dec 11

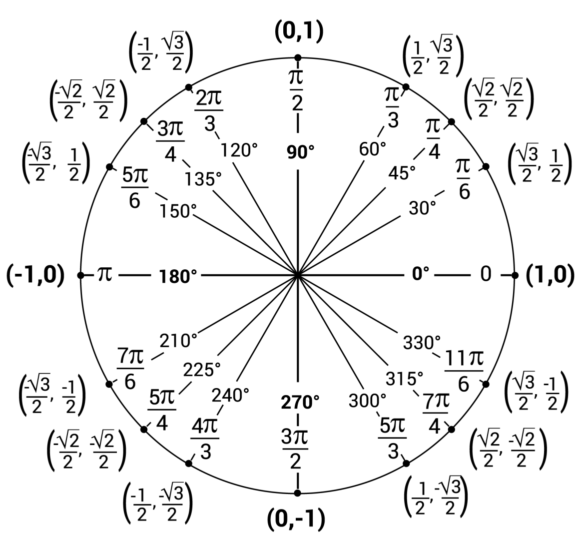
Firstname Lastname: ______________________________

UscID: ______________________________

Instructions

- This examination consists of 13 pages not including this cover page.
- Write your initials in the designated spot at the top–right of each page.
- This examination consists of 10 questions for a total of 100 points. You have 120 minutes to complete this examination.
- Do not use books, calculators, computers, tablets, or phones.
- You may use a single 8.5 in by 11 in page of notes, handwritten on both sides.
- Write legibly in the boxed area only. Cross out any work that you do not wish to have scored.
- Show all of your work and cite theorems you use. Unsupported answers may not earn credit.
- If you run out of space: there are two pages at the end where you can continue your work.
- All work you submit should represent your own thoughts and ideas. If the graders suspect otherwise: you can expect your instructor to file a report with USC's Office of Academic Integrity (OAI).

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	12	8	10	8	12	8	12	10	10	10	100



$$\sin^2\theta + \cos^2\theta = 1$$

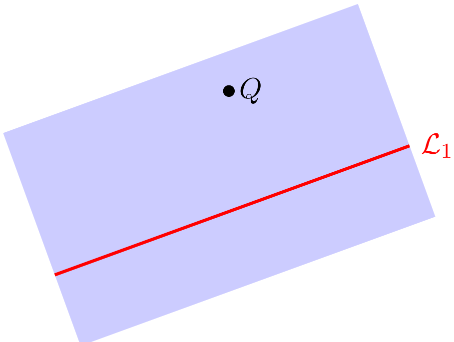
$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

1. (12 points) Consider the line $\mathcal{L}_1$ parametrized by $\mathbf{r}(t) = \langle\, t-1, 2t, -t+2\,\rangle$ and the point $Q(-1,-2,4)$.

   (a) Find an equation of the plane $\mathcal{P}$ that contains both the point Q and all points on the line $\mathcal{L}_1$.

   Note: picture not to scale.



Solution: A direction vector for $\mathcal{L}_1$ is $\mathbf{v} = \langle 1,2,-1\rangle$.

A point on the line $\mathcal{L}_1$ is $P(-1,0,2)$ and $\mathbf{PQ} = \langle 0,-2,2\rangle$. Thus a normal vector for the plane is:

$$\mathbf{n} = \mathbf{v} \times \mathbf{PQ} = \langle 2,-2,-2\rangle = 2\langle 1,-1,-1\rangle$$

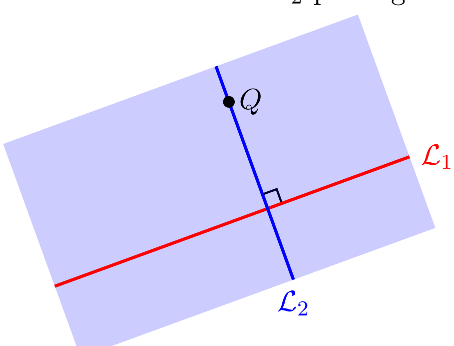
The equation for the plane is:

$$(x+1) - (y) - (z-2) = 0$$

which could be rewritten as:

$$x - y - z = -3$$

   (b) Parametrize the line $\mathcal{L}_2$ passing through the point Q and intersecting the line $\mathcal{L}_1$ orthogonally.



Solution: The point R of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$ is:

$$R = P + \mathrm{proj}_{\mathbf{v}}(\mathbf{PQ}) = (-1,0,2) + \left(\frac{\langle 1,2,-1\rangle \cdot \langle 0,-2,2\rangle}{\|\langle 1,2,-1\rangle\|^2}\right)\langle 1,2,-1\rangle = (-1,0,2) - \langle 1,2,-1\rangle = (-2,-2,3)$$

A direction vector for $\mathcal{L}_2$ is $\mathbf{RQ} = \langle 1,0,1\rangle$. A parametrization for $\mathcal{L}_2$ is:

$$\mathbf{r}(t) = R + t\mathbf{RQ} = \langle -2+t, -2, 3+t\rangle$$

2. (8 points) The function $f(x, y) = 4xy - y^2 - 2x^2y$ has **three** critical points. Find them and classify them as local minimizers, local maximizers, or saddle points.

Solution: We calculate:

$$f_x = 4y - 4xy \overset{\text{set}}{=} 0 \implies 4y(1-x) = 0$$
$$f_y = 4x - 2y - 2x^2 \overset{\text{set}}{=} 0$$

The first equation either yields $y = 0$ or $x = 1$.

If $y = 0$ then the second equation yields $4x - 2x^2 = 0 \implies 2x(2-x) = 0$ which implies $x = 0$ or $x = 2$. Thus we have critical points $(0, 0)$ and $(2, 0)$.

If $x = 1$ then the second equation yields $4 - 2y - 2 = 0 \implies y = 1$. Thus we have critical point $(1, 1)$.

The Hessian is:

$$Hf(x, y) = \begin{pmatrix} -4y & 4-4x \\ 4-4x & -2 \end{pmatrix}$$

At our three critical points we find:

$$Hf(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \quad Hf(2,0) = \begin{pmatrix} 0 & -4 \\ -4 & -2 \end{pmatrix} \quad Hf(1,1) = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}$$

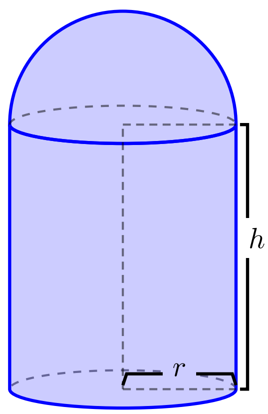
The determinants of the first two Hessians are -16 which means that $(0, 0)$ and $(2, 0)$ are saddle points.

The determinant of $Hf(1, 1)$ is 8 and the trace of $Hf(1, 1)$ is -6. This means $(1, 1)$ is a local maximizer.

3. (10 points) A factory wishes to build cylindrical bins, with a hemispherical cap on the top, and a disk at the bottom. The radius and the height of the cylindrical part are labeled by r and h respectively. The cost of the material is:

- \$1 per square feet for for the base of the bin;
- \$0.5 per square feet for the cylindrical part;
- \$1 per square feet for the hemispherical cap.

Let $f(r, h)$ be the volume of the bin and $g(r, h)$ be the total cost. If the total cost of making one bin is 28π dollars, determine the maximal volume of a single bin. Note: you may assume a global maximum exists.



Hint: Here are the relevant formulas:

[Cylinder Sides Surface Area] $= 2\pi rh$

[Cylinder Volume] $= \pi r^2 h$

[Hemisphere Surface Area] $= 2\pi r^2$

[Hemisphere Volume] $= \frac{2}{3}\pi r^3$

[Disk Area] $= \pi r^2$

Solution: The volume of the bin is:

$$f(r, h) = \pi r^2 h + \frac{2}{3}\pi r^3$$

and the total cost constraint is:

$$g(r, h) = 1\left(\pi r^2\right) + 0.5\left(2\pi rh\right) + 1\left(2\pi r^2\right) = \pi rh + 3\pi r^2 \overset{\text{set}}{=} 28\pi$$

The Lagrange equations are:

$$2\pi rh + 2\pi r^2 = \lambda(\pi h + 6\pi r) \implies 2rh + 2r^2 = \lambda(h + 6r)$$
$$\pi r^2 = \pi r\lambda \implies r = \lambda$$

I divide the first Lagrange equation by the second to obtain:

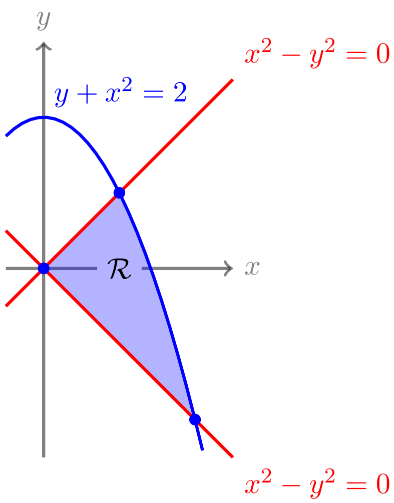
$$2h + 2r = h + 6r \implies h = 4r$$

We next substitute $h = 4r$ into the constraint equation to find:

$$4\pi r^2 + 3\pi r^2 = 28\pi \implies 7\pi r^2 = 28\pi \implies r^2 = 4 \implies r = 2$$

Thus volume is maximized when $r = 2$ feet and $h = 8$ feet. This yields $V = 32\pi + \frac{16\pi}{3} = \frac{112\pi}{3}$ cubic feet.

4. (8 points) Let $\mathcal{R}$ be the shaded region pictured below. Assume the straight lines are given by $x^2 - y^2 = 0$ and the parabola is given by $y + x^2 = 2$.



Set up **but do not evaluate** an integral (or a **sum** of integrals) over the region $\mathcal{R}$ in order $dydx$ **or** $dxdy$ (your choice: pick an order and stick with it) that **equals the area** of $\mathcal{R}$.

Solution: The two depicted points of intersection are $(1,1)$ and $(2,-2)$.

The desired integral is:

$$\int_0^1 \int_{-x}^{x} 1 \; dydx + \int_1^2 \int_{-x}^{2-x^2} 1 \; dydx$$

or:

$$\int_{-2}^{0} \int_{-y}^{\sqrt{2-y}} 1 \; dxdy + \int_0^1 \int_{y}^{\sqrt{2-y}} 1 \; dxdy$$

5. (12 points) Consider the surface $\mathcal{S}$ parametrized by:

$$\mathbf{r}(u,v) = \left\langle u\cos v, u\sin v, \frac{u^2}{2}\right\rangle$$

on the domain $\mathcal{D}$ defined by $0 \le u \le 2$ and $0 \le v \le \frac{\pi}{2}$.

(a) Compute the surface area of $\mathcal{S}$.

Solution: We find:

$$\mathbf{r}_u = \langle \cos v, \sin v, u\rangle \quad \text{and } \mathbf{r}_v = \langle -u\sin v, -u\cos v, 0\rangle$$

and:

$$\mathbf{r}_u \times \mathbf{r}_v = \langle u^2\cos v, u^2 \sin v, u\rangle$$

and:

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u^4+u^2} = u\sqrt{u^2+1}$$

The surface area is:

$$\int_0^{\frac{\pi}{2}}\int_0^2 u\sqrt{u^2+1}\, drdv = \frac{\pi}{2}\cdot\frac{1}{3}\cdot\left(5^{3/2}-1\right) = \frac{\pi}{6}\cdot\left(5^{3/2}-1\right)$$

(b) Let $f(x,y,z)$ be a function such that:

$$\begin{cases} f_x(0,2,2) = -2 \\ f_y(0,2,2) = 3 \\ f_z(0,2,2) = -1 \end{cases}$$

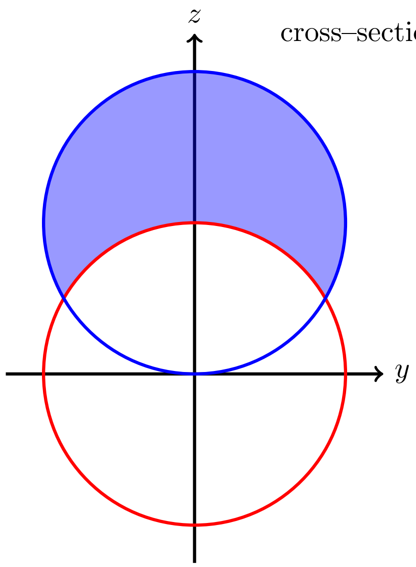
and let $(x,y,z) = \mathbf{r}(u,v)$ from the top of the page. Evaluate the following.

$$\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(2,\frac{\pi}{2})}$$

Solution: Since $(x,y,z) = \mathbf{r}(2,\frac{\pi}{2})$ we have by the chain rule:

$$\nabla f(0,2,2)\cdot \mathbf{r}_u(2,\tfrac{\pi}{2}) = \langle -2,3,-1\rangle \cdot \langle 0,1,2\rangle = 1$$

6. (8 points) Consider the solid E inside the sphere $x^2 + y^2 + (z-2)^2 = 4$ and outside the sphere $x^2 + y^2 + z^2 = 4$. Use a triple integral in spherical coordinates to calculate the volume of E.



cross–section of E with yz–plane

Solution: A spherical equation for $x^2 + y^2 + z^2 = 4$ is $\rho = 2$. A spherical equation for the other sphere is found by:

$$x^2 + y^2 + (z-2)^2 = 4 \implies x^2 + y^2 + z^2 = 2z \implies \rho^2 = 2\rho\cos\phi \implies \rho = 4\cos\phi$$

So the bounds on ρ are $2 \leq \rho \leq 4\cos\phi$. The spheres intersect when:

$$2 = 4\cos\phi \implies \phi = \frac{\pi}{3}$$

So the bounds on ϕ are $0 \leq \phi \leq \frac{\pi}{3}$. Lastly the bounds on θ are $0 \leq \theta \leq 2\pi$.

The integral we want is:

$$\iiint_E dV = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_2^{4\cos\phi} \rho^2 \sin\phi \; d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \frac{64}{3}\cos^3\phi\sin\phi - \frac{8}{3}\sin\phi \; d\phi d\theta = \cdots$$

$$\cdots = \int_0^{2\pi} \frac{16}{3}\left(1 - \frac{1}{16}\right) + \frac{8}{3}\left(\frac{1}{2} - 1\right) \; d\theta = \int_0^{2\pi} \frac{11}{3} \; d\theta = \frac{22\pi}{3}$$

7. (12 points) Consider the vector field $\mathbf{F}(x,y) = \Big[\ln(y) + axy^3\Big]\,\mathbf{i} + \Big[(a+1)x^2y^2 + \frac{x}{y}\Big]\,\mathbf{j}$

(a) Find the value of the constant a for which $\mathbf{F}$ is conservative in the open upper half-plane $\{y > 0\}$. You must justify that your answer is correct.

Solution: The upper half–plane is simply–connected, so it suffices to find the value of a for which the curl of $\mathbf{F}$ is the zero vector field. We compute:

$$\operatorname{curl}\mathbf{F} = \left[\left((2a+2)xy^2 + \frac{1}{y}\right) - \left(\frac{1}{y} + 3axy^2\right)\right]\mathbf{k} = (2-a)xy^2\mathbf{k} \overset{\text{set}}{=} 0 \implies a = 2$$

(b) For the value of a found above, find a potential f for $\mathbf{F}$. Note: recall that a potential for a vector field is a function whose gradient equals that vector field.

Solution: We find:

$$f(x,y) = \int \ln(y) + 2xy^3\;dx = x\ln(y) + x^2y^3 + g(y)$$

and then:

$$f_y(x,y) = \frac{x}{y} + 3x^2y^2 + g'(y) \overset{\text{set}}{=} 3x^2y^2 + \frac{x}{y} \implies g'(y) = 0 \implies g(y) = C$$

So every potential has the form:

$$f(x,y) = x\ln(y) + x^2y^2 + C \quad \text{where } C \text{ is a constant}$$

(c) Evaluate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ where $\mathcal{C}$ is the curve parametrized by:

$$\mathbf{r}(t) = \langle\, 2+\cos t, 1+\sin t\,\rangle \quad \text{with } 0 \le t \le \pi$$

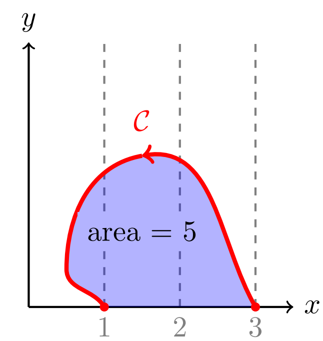
Solution: We calculate using our potential:

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(1,1) - f(3,1) = (0+1) - (0+9) = -8$$

8. (10 points) Consider the 2–dimensional vector field:

$$\mathbf{F}(x,y) = (e^y + 3y)\,\mathbf{i} + (xe^y + 5x)\,\mathbf{j}$$

Let $\mathcal{C}$ be the oriented curve depicted below and assume the area of the shaded region is 5. Note: the shaded region is bounded by both $\mathcal{C}$ and the line segment from $(1,0)$ to $(3,0)$.



Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. Hint: extend $\mathcal{C}$ to a closed curve and involve Green's Theorem.

Solution: Let D be the enclosed region and $\mathcal{C}'$ be the line segment from $(1,0)$ to $(3,0)$.

Then the curve [$\mathcal{C}$ followed by $\mathcal{C}'$] is oriented so the enclosed region is on the left. So we set up Green's Theorem:

$$\int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_D (e^y + 5) - (e^y + 3)\ dA = \iint_D 2\ dA = 2 \cdot 5 = 10$$

And next using the parametrization $\mathbf{r}(x) = \langle x, 0 \rangle$ with $1 \le x \le 3$ of $\mathcal{C}'$ we calculate:

$$\int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \langle 1, \star \rangle \cdot \langle 1, 0 \rangle dx = \int_1^3 1\ dx = 2$$

Consequently:

$$2 + \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 10 \implies \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 8$$

9. (10 points) Let $\mathcal{S}$ be the **closed** surface consisting of the portion of the cone $z = \sqrt{x^2+y^2}$ with $0 \le z \le 2$ along with its **top** at $z = 2$. Orient $\mathcal{S}$ with **inward** normals and evaluate the following integral by using the divergence theorem.

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \quad \text{where} \quad \mathbf{F}(x,y,z) = \left(\tfrac{5}{3}x^3 + ze^{y^3}\right)\mathbf{i} + \left(y + z^2\cos(x^2)\right)\mathbf{j} + \left(1 - z\right)\mathbf{k}$$

Solution: We use the divergence theorem but note that inward normals is the opposite orientation of the one compatible with the divergence theorem. We compute:

$$\operatorname{div}\mathbf{F} = 5x^2 + 1 - 1 = 5x^2$$

The region E enclosed by $\mathcal{S}$ can be described in cylindrical as $r \le z \le 2$ and $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

The divergence theorem says:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = -\iiint_E 5x^2 \, dV = -\int_0^{2\pi}\int_0^2\int_r^2 5r^3\cos^2\theta \, dz dr d\theta = -\int_0^{2\pi}\int_0^2 10r^3\cos^2\theta - 5r^4\cos^2\theta \, dr d\theta = \cdots$$

$$\cdots = -\int_0^{2\pi} 40\cos^2\theta - 32\cos^2\theta \, d\theta = -\int_0^{2\pi} 8\cos^2\theta \, d\theta = -\int_0^{2\pi} 4 + 4\cos 2\theta \, d\theta = -8\pi$$

10. (10 points) Let $\mathcal{C}$ be the **boundary** curve of the portion of the **surface** $z = xy$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Orient $\mathcal{C}$ **clockwise** as viewed from above. Evaluate the following **line** integral by using Stokes's Theorem to convert to an appropriate and simpler **surface** integral.

$$\int_{\mathcal{C}} \left[e^{\sin(e^x)} + z^2\right] dx + \left[\frac{1}{1+y^4}\right] dy + \left[1 + y^2\right] dz$$

Solution: Let $\mathbf{F}$ be the vector field being integrated. Then:

$$\operatorname{curl} \mathbf{F} = \langle 2y, 2z, 0 \rangle$$

Let $\mathcal{S}$ be the surface $z = xy$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. To have a compatible orientation with the clockwise orientation of its boundary $\mathcal{C}$, the surface $\mathcal{S}$ should be oriented with **downward** normals.

If we parametrize $\mathcal{S}$ with parameters x and y then:

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -y, -x, 1 \rangle$$

This is upwards, so we negate it to obtain downwards:

$$\langle y, x, -1 \rangle$$

We next use Stokes's Theorem:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \langle 2y, 2z, 0 \rangle \cdot d\mathbf{S} = \int_0^1 \int_0^1 \langle 2y, 2xy, 0 \rangle \cdot \langle y, x, -1 \rangle dydx = \cdots$$

$$\cdots = \int_0^1 \int_0^1 2y^2 + 2x^2 y \; dydx = \frac{2}{3} + \frac{1}{3} = 1$$

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