Example 1. Let $\vec{v}$ and $\vec{w}$ be **unit** vectors so that $\vec{v} \cdot \vec{w} = \frac{1}{2}$ and find $\|2\vec{v} + \vec{w}\|^2$.

Solution. First we use that $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ to write:

$\|2\vec{v} + \vec{w}\|^2 =$

Next we use **distributivity** of dot product over addition.

$\cdots =$

Next we **commutativity** of dot products and of scalar multiplication.

$\cdots =$

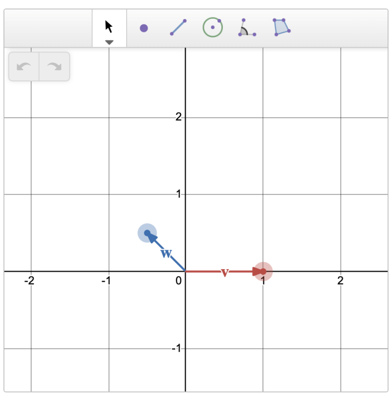
Next we use that $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

$\cdots =$

Lastly we use that $\vec{v}$ and $\vec{w}$ are unit vectors and that $\vec{v} \cdot \vec{w} = \frac{1}{2}$.

$\cdots =$

A. **Dot Products and Angles.** We explore further the meaning of dot products.



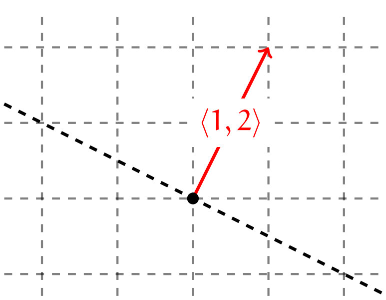
Let $\vec{v}$, $\vec{w}$ be vectors with angle θ between them. Then:

$\vec{v} \cdot \vec{w} =$

and thus $\vec{v} \perp \vec{w}$ if and only if:

Example 2. Find all vectors orthogonal to $\vec{v} = \langle a, b \rangle$ with the same length as $\vec{v}$.

Solution. First we build intuition by considering specific example $\vec{v} = \langle 1, 2 \rangle$.



Now we generalize this to $\vec{v} = \langle a, b \rangle$ by selecting:

which we confirm are $\perp$ to $\vec{v}$ by verifying:

Example 3. Find the angle between $\vec{v} = \langle 1, 0, 1 \rangle$ and $\vec{w} = \langle -1, 1, 0 \rangle$.

Solution. First we compute:

$\vec{v} \cdot \vec{w} =$

$\|\vec{v}\| \|\vec{w}\| \cos\theta =$

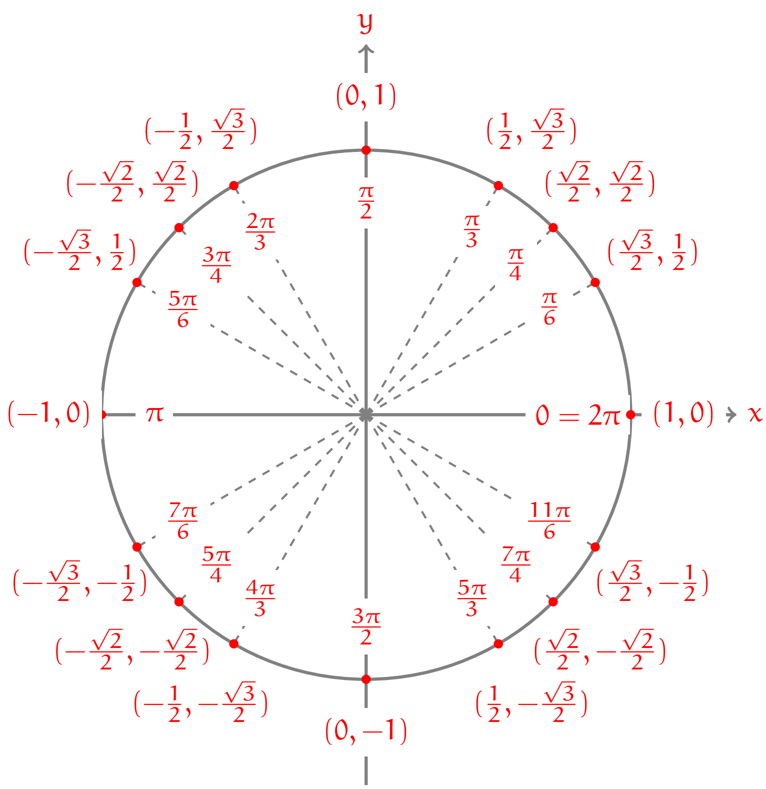
These are equivalent expressions for the dot product, and thus:

Now we may use the unit circle to find the value of θ.

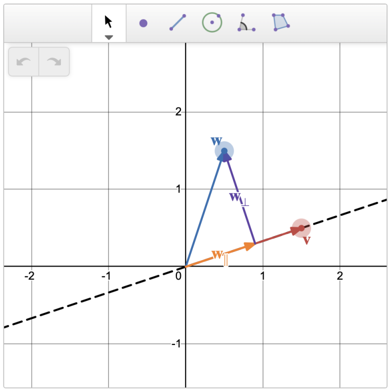
$\theta =$

Or we may leave our answer in terms of inverse trig:

$\theta =$



B. **Projections.** Next we explore orthogonal projections.



Let $\vec{v}$ and $\vec{w}$ be vectors, where $\vec{v}$ is nonzero.

The **orthogonal projection** of $\vec{w}$ along $\vec{v}$ is:

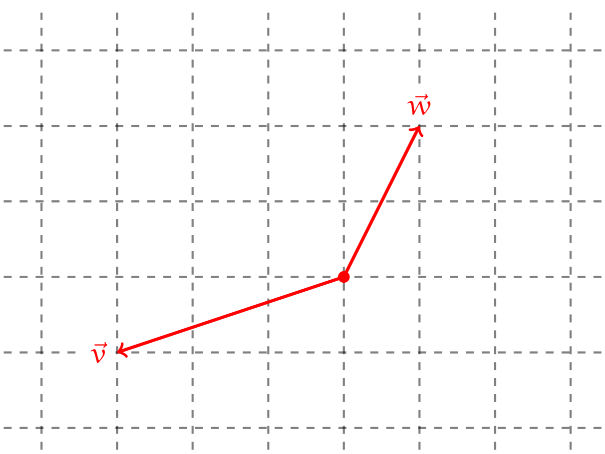
$\vec{w}_{||} = \text{proj}_{\vec{v}}(\vec{w}) =$

and the **scalar component** of $\vec{w}$ along $\vec{v}$ is:

$\pm\|\vec{w}_{||}\| = \text{comp}_{\vec{v}}(\vec{w}) =$

with a negative if $\vec{w}_{||}$ is opposite $\vec{v}$.

Example 4. Find the orthogonal projection of $\vec{w} = \langle 1, 2\rangle$ onto $\vec{v} = \langle -3, -1\rangle$ and find the scalar component of $\vec{w}$ along $\vec{v}$.



$\text{comp}_{\vec{v}}(\vec{w}) =$

$\vec{w}_{||} = \text{proj}_{\vec{v}}(\vec{w}) =$