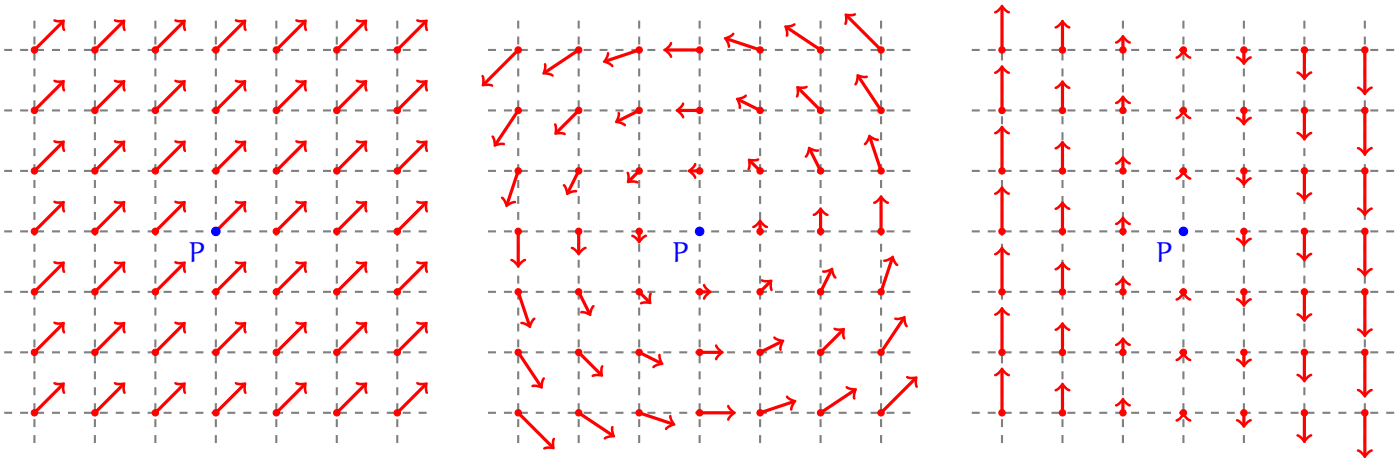
Example 1. For each vector field $\vec{F}$, use the righthand rule for circulation to decide whether $\operatorname{curl}\vec{F}(P)$ is $\vec{0}$, points into the page, or points out of the page.



A. **Curl and Conservativity.** Conservative vector fields don't feature circulation.

Every conservative vector field is **irrotational**, meaning: $\operatorname{curl}\vec{F} =$

This can be proven by direct calculation.

Let us examine this condition for a two–dimensional vector field.

$\vec{F}(x,y) =$

$\operatorname{curl}\vec{F}(x,y) =$

Thus a two–dimensional vector field $\langle P, Q\rangle$ is irrotational if and only if:

We use this idea to show that the following vector field is not conservative.

$\vec{F}(x,y) = \langle -y, x\rangle$

We compute

$Q_x =$

$P_y =$

These are unequal, thus $\vec{F}$ is:

hence is:

Example 2. Show the following vector field is irrotational, but not conservative.

$$\vec{F}(x,y) = \frac{-y\vec{\imath} + x\vec{\jmath}}{x^2 + y^2}$$

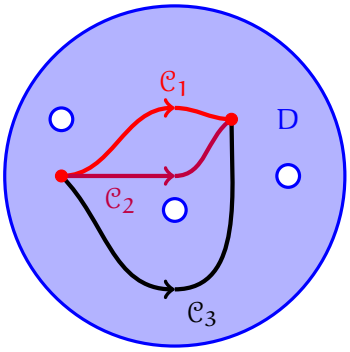
Soluton. Last class we showed this vector field was not conservative by finding

So it remains to show it is irrotational.

We use the shortcut formula $\text{curl}\langle P, Q\rangle = (Q_x - P_y)\vec{k}$

B. **Simple–Connectedness.** Let $\vec{F}$ be irrotational on a region D.

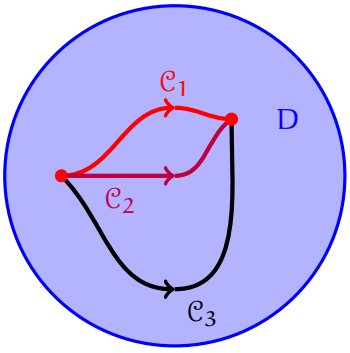
Consider curves in the region D with a common start and common end.



They are **path–homotopic** if, with endpoints fixed and within region D, they can

If $\vec{F}$ is irrotational on region D, then $\vec{F}$ integrates to the same value over

Roughly, a 2D region with no holes is said to be:



and has the feature that all paths with a given start and end are:

Theorem. If $\vec{F}$ is irrotational on a region that is:

then $\vec{F}$ will have:

which turns out to be enough to guarantee the $\vec{F}$ on that region is:

Example 3. Let $\mathcal{C}$ be an arbitrary closed curve in the xy–plane. Show

$$\oint_{\mathcal{C}} e^{\cos(e^x)}\, dx + \sin(\sin(e^{\sin y}))\, dy = 0$$

Solution. This is an integral of the vector field

$\vec{F}(x, y) =$

The domain of this particular vector field is

Further $\vec{F}(x, y)$ is irrotational on its domain because

$\operatorname{curl} \vec{F}(x, y) =$

As $\vec{F}$ is irrotational on a simply–connected region, the previous theorem implies

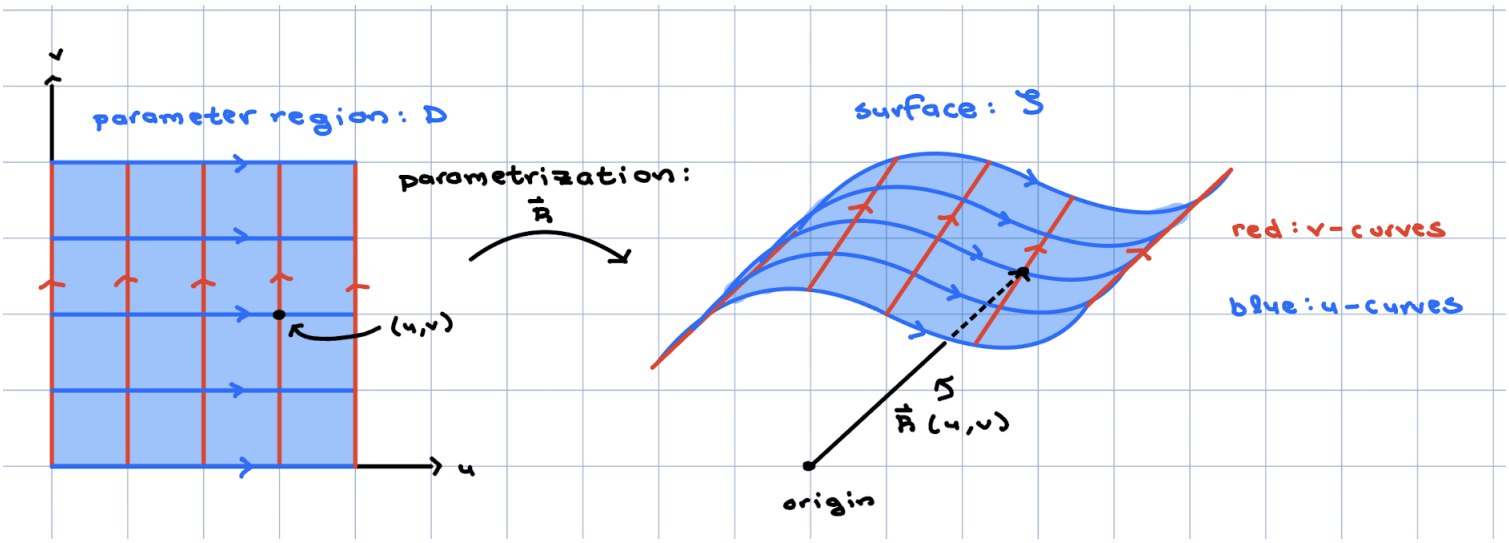
Conservative vector fields integrate over closed curves to:

C. **Parametrized Surfaces.** We parametrize a surface S.

We need two parameters:

and a parameter region:

and a **parametrization** function $\vec{R}(u, v)$ that outputs exactly

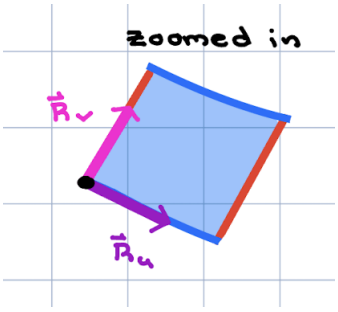


We can take partials derivatives of our parametrization to obtain tangent vectors.

[u–tangent vector to surface] =

[v–tangent vector to surface] =

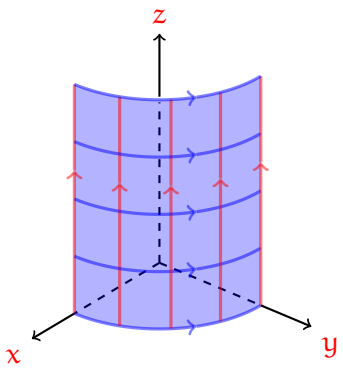
[normal vector to surface] =



In the sketched surface at the top, the u–direction cross v–direction normals

A choice of direction for normals to the surface is called an

Example 4. Consider S given by $x^2 + y^2 = 1$ with $0 \leqslant z \leqslant 2$ and $x \geqslant 0$ and $y \geqslant 0$.



Parametrize S and find the orientation induced by your parametrization.

Solution. The surface S is readily described in cylindrical coordinates as

Thus it would be reasonable to select parameters:

Bsaed on the sketch, the cross of the θ– and z–directions points

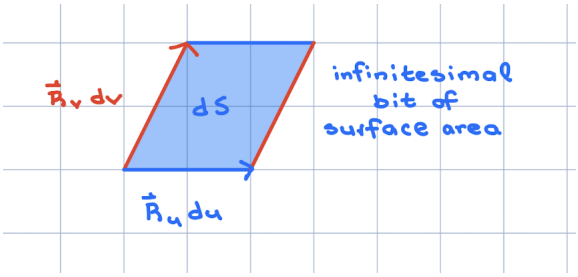
We may also verify this computationally.

$\vec{R}_\theta \times \vec{R}_z =$

D. **Surface Area.** Let $\mathcal{S}$ be parametrized by $\vec{R}(u, v)$.

Let dS denote an infinitisemal bit of surface area from S. Its dimensions are

- an infinitesimal change in the u–direction:
- an infinitesimal change in the v–direction:



The area of a parallellogram given by two vectors is the length of their cross.

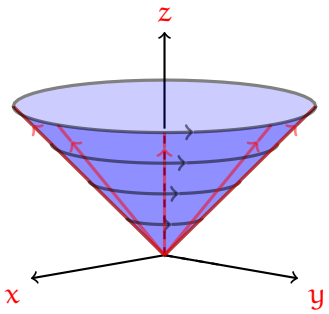
$dS =$

We compute the total surface area by adding dS at all points of the surface.

Let $\mathcal{S}$ be a surface. Then

[surface area of $\mathcal{S}$] $=$

Example 5. Find the surface area of the portion of $z = \sqrt{x^2 + y^2}$ with $0 \leqslant z \leqslant 2$.



Solution. The cone is readily described in cylindrical as

Thus it would be reasonable to select parameters:

We temporarily forget the cone, and instead calculate dS in the **general** scenario

In such a scenario we have

$\vec{R}(r, \theta) =$

$dS =$

We specialize to our cone, defined by equation $z = r$, to obtain

$dS =$

Next we compute the surface area.

[surface area] $=$