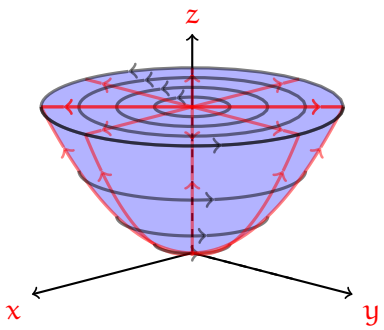
A. **Scalar Surface Integrals.** We integrate $f\ dS$ over a surface S.

The **scalar surface integral** of function $f(x,y,z)$ over S is

Example 1. Let S be the surface consisting of $z = x^2 + y^2$ with $0 \leqslant z \leqslant 1$ along with its top at $z = 1$.



Find $\iint_S 1 + z - x^2 - y^2\ dS$.

Solution. The function we will integrate is $f(x,y,z) = 1 + z - x^2 - y^2$.

Our surface consists of two parts, and thus

$\iint_S f\ dS =$

First we address the **sides**, which are readily described in cylindrical as

So naturally we parametrize the sides with:

We use the shortcut from the recordings for dS for surfaces $z = \text{function}(r)$.

$dS =$

For the sides, where:

$dS =$

And thus ... continued on next page ...

... continued from last page ...

$$\iint_{\text{sides}} 1 + z - x^2 - y^2 \; dS =$$

Next we address the **top**, which in cylindrical is

So naturally we parametrize with:

We reuse the shortcut $dS = r\sqrt{\left[\frac{\partial z}{\partial r}\right]^2 + 1}\; dr d\theta$ for $z = \text{function}(r)$.

For the top, where $z = 1$, we have

$dS =$

And thus

$$\iint_{\text{top}} 1 + z - x^2 - y^2 \; dS =$$

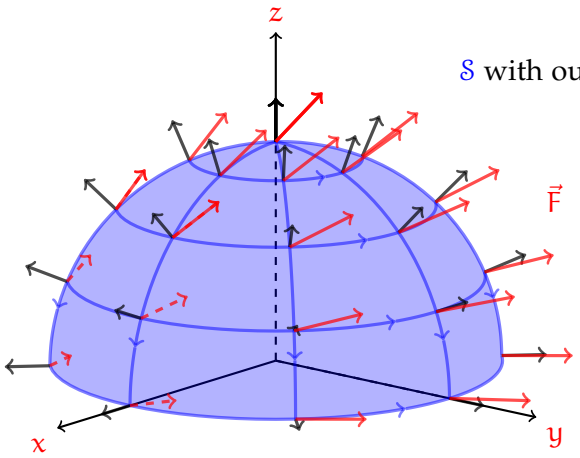
The final answer is the sum of the integrals over the sides and top

$$\iint_S f \; dS =$$

B. **Vector Surface Integrals.** We can also integrate vector fields over surfaces.

Let $\mathcal{S}$ be an **oriented** surface $\mathcal{S}$, i.e. a surface with a choice of

Consider $\mathcal{S}$ as a mesh through which fluid flows according to vector field $\vec{F}$.

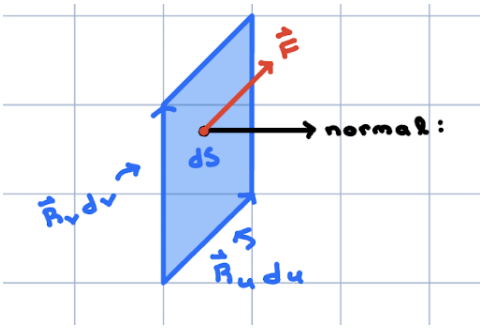


We will measure the net rate of fluid flow across the surface.

Parametrize the surface using $\vec{R}(u, v)$ such that the **infinitesimal normal**

$d\vec{S} =$

points in the direction of the given:



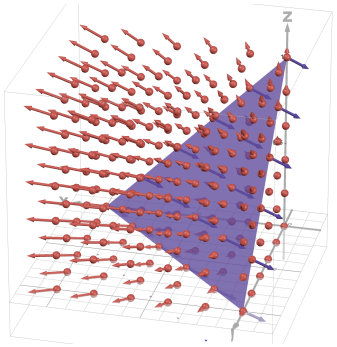
The rate at which fluid flows across dS in the direction of the normal is

We sum across all points on the surface to obtain the net flow across the surface.

The **flux** (or **vector surface integral**) of $\vec{F}$ across surface $\mathcal{S}$ is

[rate of fluid flow across $\mathcal{S}$] =

Example 2. Find the flux of $\vec{F}(x, y, z) = \langle 2x, y, z \rangle$ across the surface S equal to the portion of the plane by $x + y + z = 1$ in the first octant, and oriented with downward normals.



Solution. The surface is readily described using parameters x and y.

We temporarily consider the general scenario of parameters x and y.

$\vec{R}(x, y) =$

$d\vec{S} =$

We specialize to our case, where $z = 1 - x - y$, and adjust for downwards normals.

$d\vec{S} =$

Next we compute the flux by writing entirely in terms of our parameters x and y.

$$\iint_S \langle 2x, y, z \rangle \cdot d\vec{S} =$$