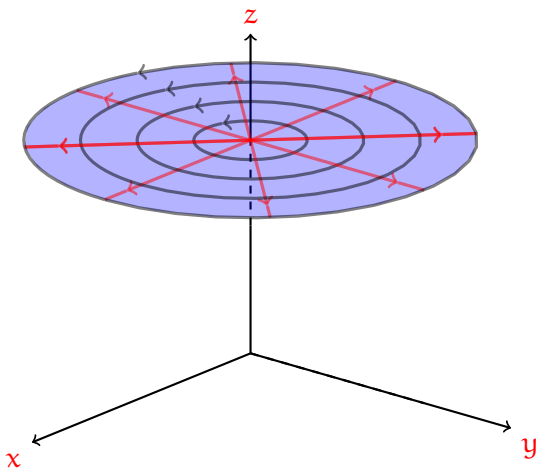
Example 1. S is the surface $x^2+y^2 \leqslant 1$ with $z = 1$ oriented with upward normals.



Let $\vec{F}$ be a vector field which, at points in S,

- is at angle $\pi/3$ from the upwards normals to S
- has length $\|\vec{F}(x,y,z)\| = 3 + z^2$

Find $\iint_S \vec{F} \cdot d\vec{S}$.

Solution. While r and θ is also fine, we parametrize S with:

We use the shortcut for upwards $d\vec{S} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\rangle \, dA$ for parameters x and y.

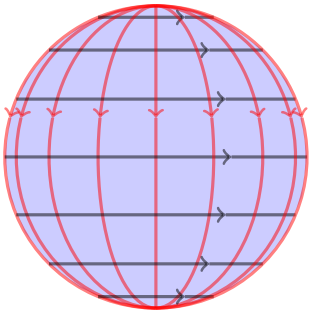
In our case where $z = 1$ we have

$d\vec{S} =$

Lastly we use the rule $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\| \cos\theta$ to obtain

$\iint_S \vec{F} \cdot d\vec{S} =$

Example 2. Let S be the sphere $x^2 + y^2 + z^2 = 4$ oriented with outward normals.



Find $\iint_S \langle x, y, z\rangle \cdot d\vec{S}$.

Solution. Our sphere is naturally parametrized by:

Let's temporarily consider the general scenario of a sphere $\rho = R$.

$\vec{R}(\phi, \theta) = \langle x, y, z\rangle =$

$d\vec{S} =$

We specialize to our sphere $\rho = 2$.

$d\vec{S} =$

Next we return to our integral.

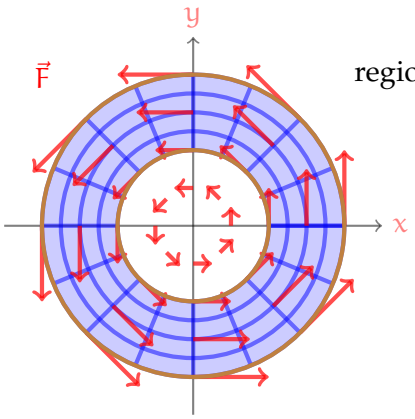
$\iint_S \langle x, y, z\rangle \cdot d\vec{S} =$

A. **Green's Theorem.** We connect double integrals and vector line integrals.

Consider a compact two–dimensional region D in the xy–plane.

Denote by ∂D its **boundary** aka its:

Consider also a vector field $\vec{F} = \langle P, Q \rangle$.



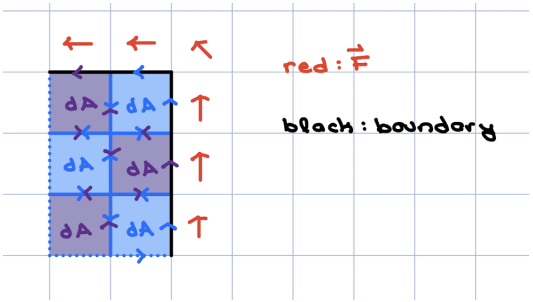
region D with boundary ∂D

Observe that the boundary is made up of curves that are

•

• not–self–intersecting aka:

We divide our region into infinitesimal bits of area.



$\operatorname{curl} \vec{F} = (Q_x - P_y)\vec{k}$ will compute the circulation about dA in the sense that

[counterclockwise–measured–positive circulation about dA] =

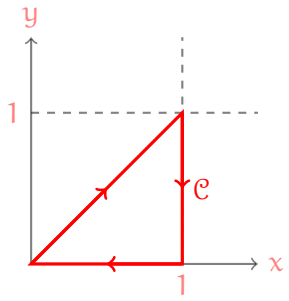
When summed over all dA's, flow calculations on common edges of dA's will

Green's Theorem. Consider the setup given above. Then

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} =$$

where ∂D is oriented counterclockwise about enclosed region.

Example 3. Let $\mathcal{C}$ be the closed curve depicted below, with clockwise orientation.



Use Green's Theorem to find $\oint_{\mathcal{C}} y(1-e^{x^2})\,dx + x\,dy$.

Solution. The vector field in consideration here is $P\vec{\imath} + Q\vec{\jmath} = y(1-e^{x^2})\vec{\imath} + x\vec{\jmath}$.

The curve $\mathcal{C}$ completely encloses the region D given by

But $\mathcal{C}$ goes clockwise about enclosed region, so we adjust Green's Theorem

$\oint_{\mathcal{C}} y(1-e^{x^2})\,dx + x\,dy =$