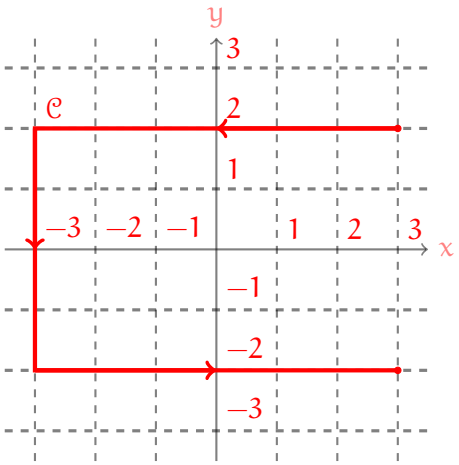
Example 1. Let $\vec{F} = \langle x^2 + y^2, 2xy \rangle$ and $\mathcal{C}$ be the counterclockwise curve below.



Use Green's Theorem to find $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$.

Solution. Green's Theorem requires a curve that entirely encloses a region.

Our $\mathcal{C}$ does not. So we close the curve by introducing the curve $\mathcal{C}'$ given by

The region that is counterclockwise–enclosed by $\mathcal{C}$ and $\mathcal{C}'$ is D given by

We may now apply Green's Theorem $\oint_{\partial D} \langle P, Q \rangle \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$.

then rewriting

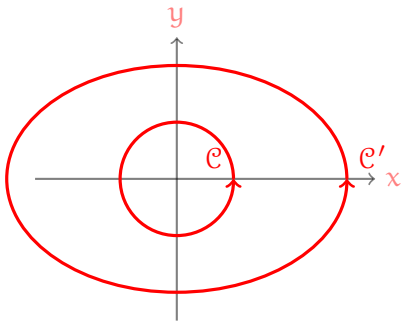
Example 2. Let $\vec{F}(x, y)$ be defined an irrotational everywhere except the origin.

For the curve $\mathcal{C}$ below you are given

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 2\pi$$

For the curve $\mathcal{C}'$ depicted below find

$$\oint_{\mathcal{C}'} \vec{F} \cdot d\vec{r} = ?$$



Solution. Observe that $\mathcal{C}$ and $\mathcal{C}'$ together form the boundary of a region D.

In order for bits of region to be counter–clockwise enclosed we assign orientations

By Green's Theorem we have

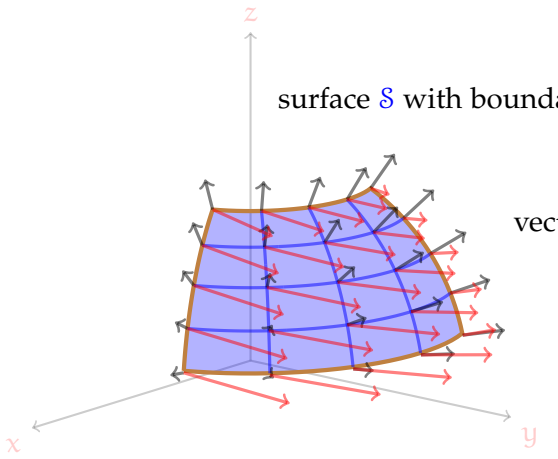
But $\vec{F}$ is irrotational throughout D, hence:

So we may continue.

A. **Stokes's Theorem.** We extend Green's Theorem to surfaces.

Let's consider an oriented surface $\mathcal{S}$ in xyz–space and its **boundary** $\partial\mathcal{S}$.

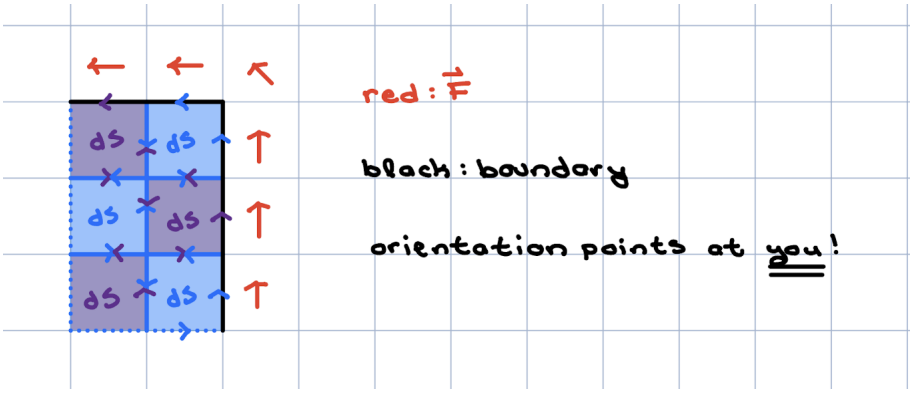
Imagine the surface in the presence of a vector field $\vec{F}$.



surface $\mathcal{S}$ with boundary $\partial\mathcal{S}$ and out–of–page normals

vector field $\vec{F}$

We divide the region into infinitesimal bits of surface area.



curl $\vec{F}$ will compute the circulation about dS in the sense that

[cc–measured–positive when–viewed–so–$d\vec{S}$–points–at–you circulation about dS] =

When summed over all dS's, flow calculations on common edges of dS's will

Stokes's Theorem. Consider the setup given above. Then

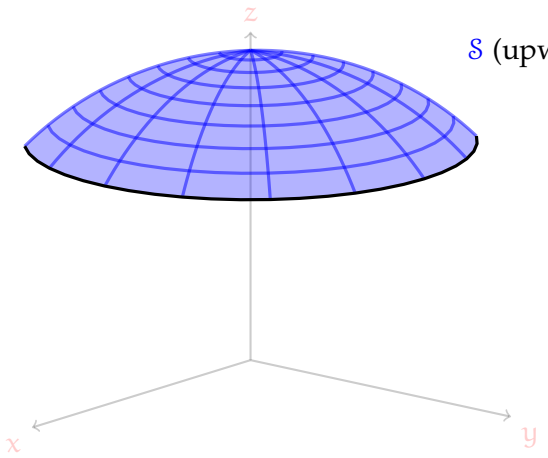
$$\oint_{\partial\mathcal{S}} \vec{F} \cdot d\vec{r} =$$

where ∂S is oriented cc about enclosed surface, when $d\vec{S}$ points at you.

Example 3. Let

$$\vec{F} = -y\vec{\imath} + \left[x + (z-1)xe^x\right]\vec{\jmath} + \left[x^2 + y^2\right]\vec{k}$$

and S be spherical cap $x^2 + y^2 + z^2 = 2$ with $z \geqslant 1$ and oriented upwards.



S (upward normals)

Use Stoke's Theorem to find $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$.

Solution. Our boundary ∂S is given by

We are given upwards normals. So a compatible orientation for ∂S is

We parametrize ∂S in agreement with this orientation as

By Stokes's Theorem

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} =$$