Math 226 Final Exam

Sp26

Sat May 9

Firstname Lastname: ______________________________

USC Email: ______________________ UscID: ______________________

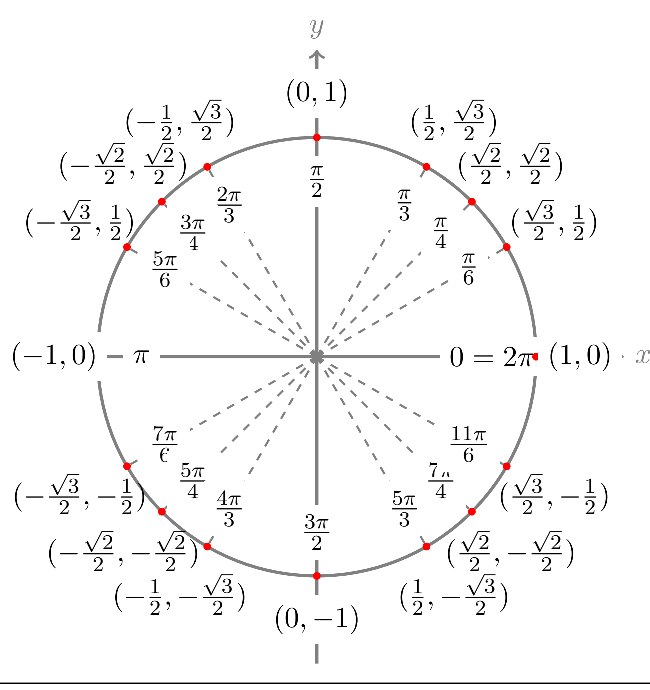
Instructions.

- This examination consists of 20 pages not including this cover page.
- Write your initials in the designated spot on each page that contains work which you would like to have graded.
- This examination consists of 9 questions for a total of 100 points. You have 120 minutes to complete it.
- Do not use books, calculators, computers, tablets, or phones.
- You may use a single 8.5 in by 11 in page of notes, handwritten on both sides.
- Write legibly in the boxed area only. Cross out any work that you do not wish to have scored. Be sure to indicate your final answer, for example by boxing it.
- Show all of your work and cite theorems you use. Unsupported answers may not earn credit.
- If you run out of space: there is an extra page after each problem, and one extra page at the end. Please indicate on the page containing the original question when you are continuing your work on another page.
- All work you submit should represent your own thoughts and ideas. If the graders suspect otherwise, you can expect your instructor to file a report with USC's Office of Academic Integrity.

Circle your instructor: Prof Alcala Prof Geske Prof Reyes Souto Prof Sethi

---

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	12	12	10	12	12	12	100



1. (10 points) Defined below are the planes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and the line ℓ in xyz–space.

$$\mathcal{P}_1 \colon x + 2y - z = 4 \qquad \mathcal{P}_2 \colon 2x + 4y - 2z = 4 \qquad \mathcal{P}_3 \colon 2x + 4y - z = 4 \qquad \ell : \langle 1 + 2t, 2 - t, 1 \rangle$$

Note: two non–parallel planes in xyz–space are guaranteed to intersect.

(a) Decide whether $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel. Then find the (shortest) distance between them.

(b) Decide whether $\mathcal{P}_1$ and $\mathcal{P}_3$ are parallel. Then find the (shortest) distance between them.

(c) Decide whether $\mathcal{P}_3$ and ℓ are parallel. Then find the (shortest) distance between them.

Solution: (a) We rescale the equation defining $\mathcal{P}_2$ to be $x + 2y - z = 2$. Thus the planes are parallel with common normal $\mathbf{n} = \mathbf{n}_1 = \mathbf{n}_2 = \langle 1, 2, -1 \rangle$.

Using any points P_1 and P_2 on the two planes, the shortest distance between the planes is thus:

$$\left| \frac{\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_2)}{\|\mathbf{n}\|} \right| = \frac{4-2}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

(b) A normal to the first plane is $\mathbf{n}_1 = \langle 1, 2, -1 \rangle$. A normal to the third plane is $\mathbf{n}_3 = \langle 2, 4, -1 \rangle$. The normals are not parallel, and thus the planes are not parallel, so intersect. The shortest distance between the planes is thus 0.

(c) A direction vector for the line is $\mathbf{v} = \langle 2, -1, 0 \rangle$. A normal for the third plane is $\mathbf{n}_3 = \langle 2, 4, -1 \rangle$. Since:

$$\mathbf{n}_3 \cdot \mathbf{v} = \langle 2, 4, -1 \rangle \cdot \langle 2, -1, 0 \rangle = 4 - 4 + 0 = 0$$

it follows that the line and plane are parallel. So the shortest distance between the line and the plane is the same as the shortest distance between any point on the line and the plane. Using the point $P(1, 2, 1)$ and any point P_3 on the plane, we calculate:

$$\left| \frac{\mathbf{n}_3 \cdot (\mathbf{p} - \mathbf{p}_3)}{\|\mathbf{n}_3\|} \right| = \frac{2 + 8 - 1 - 4}{\sqrt{21}} = \frac{5}{\sqrt{21}}$$

You may continue your work for Q1 on this page.

2. (10 points) The temperature in $^\circ C$ at position (x, y, z) measured in meters relative to the center of a heater is

$$T(x, y, z) = 150e^{-x^2-2y^2-z^2}$$

(a) Find the rate of change of temperature at the point $P(0, 2, -2)$ in the direction from P to $Q(1, -1, 1)$.

Hint: ensure you use a **unit** direction in your calculations.

(b) In which **unit** direction from P does temperature increase fastest? What is that rate of increase?

Solution: (a) We calculate:

$\mathbf{q} - \mathbf{p} = \langle 1, -3, 3\rangle$

and then normalize to obtain unit direction:

$\mathbf{u} = \dfrac{1}{\sqrt{19}} \langle 1, -3, 3\rangle.$

The gradient of T is:

$\nabla T(x, y, z) = 300e^{-x^2-2y^2-z^2} \langle -x, -2y, -z\rangle \implies \nabla T(0, 2, -2) = 300e^{-12}\langle 0, -4, 2\rangle = 600e^{-12}\langle 0, -2, 1\rangle$

The relevant directional derivative is:

$\nabla T(0, 2, -2) \cdot \mathbf{u} = \dfrac{600e^{-12}}{\sqrt{19}} \cdot (0 + 6 + 9) = \dfrac{5400e^{-12}}{\sqrt{19}}$ degrees per meter

(b) The unit direction from P in which temperature increases fastest is the normalized gradient of T at P:

$\dfrac{600e^{-12}\langle 0, -2, 1\rangle}{\|600e^{-12}\langle 0, -2, 1\rangle\|} = \dfrac{\langle 0, -2, 1\rangle}{\|\langle 0, -2, 1\rangle\|} = \dfrac{1}{\sqrt{5}}\langle 0, -2, 1\rangle$

The maximal rate of temperature increase is the magnitude of the gradient of T at P:

$\|600e^{-12}\langle 0, -2, 1\rangle\| = 600\sqrt{5}e^{-12}$ degrees per meter

You may continue your work for Q2 on this page.

3. (10 points) Use Lagrange multipliers to find the **minimum** value of

$$f(x, y, z) = 2x^2 + xy + 2xz - 6x - 5y + z^2 - 2z$$

subject to $x + y + 2z = 1$. You may assume that an absolute minimum exists.

Solution: We set up the system:

$$\begin{cases} 4x + y + 2z - 6 = \lambda \\ x - 5 = \lambda \\ 2x + 2z - 2 = 2\lambda \implies x + z - 1 = \lambda \end{cases}$$

Eliminating λ from the equations yields system:

$$\begin{cases} 4x + y + 2z - 6 = x - 5 \\ x - 5 = x + z - 1 \implies z = -4 \end{cases}$$

Substituting the second equation into the first gives:

$4x + y - 8 - 6 = x - 5 \implies y = -3x + 9$

Writing the constraint entirely in terms of x gives:

$x - 3x + 9 - 8 = 1 \implies x = 0 \implies y = 9$

The minimum value is:

$f(0, 9, -4) = -45 + 16 + 8 = -21$

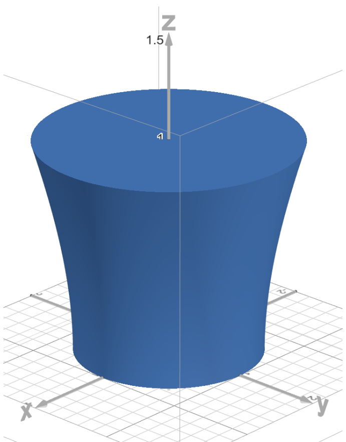
You may continue your work for Q3 on this page.

4. (12 points) The one–sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ with $0 \le z \le 1$ bounds a solid E where the coordinates are measured in cm. The density of mass of E is constant:

   $\delta = 24 \text{ g/cm}^3$

   Locate the center of mass of E. You may use that the total mass of E is 32π g.

   Hint: Don't compute the total mass by hand, as it has been provided to you. You should be able to quickly determine the x–coordinate and y–coordinate of the center of mass using symmetry. Cylindrical integration in the order $drdzd\theta$ may be suitable for finding the z–coordinate.



Solution: In cylindrical coordinates, the hyperboloid is defined by:

$r^2 - z^2 = 1 \implies r = \sqrt{1+z^2}$

Therefore the solid E is defined by:

$0 \le r \le \sqrt{1+z^2}$

$0 \le z \le 1$

$0 \le \theta \le 2\pi$

The total mass is:

$$\int\int\int_E 24\ dzdydx = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1+z^2}} 24r\ drdzd\theta = \int_0^{2\pi}\int_0^1 12 + 12z^2\ dzd\theta = \int_0^{2\pi} 16\ d\theta = 32\pi$$

By symmetry: the x–coordinate and y–coordinate of the center of mass are $\overline{x} = 0$ and $\overline{y} = 0$.

To get the z–coordinate of the center of mass we integrate:

$$\int\int\int_E 24z\ dzdydx = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1+z^2}} 24zr\ drdzd\theta = \int_0^{2\pi}\int_0^1 12z + 12z^3\ dzd\theta = \int_0^{2\pi} 9\ d\theta = 18\pi$$

Therefore the z–cordinate of the center of mass is:

$$\overline{z} = \frac{18\pi}{32\pi} = \frac{9}{16}$$

The center of mass is thus:

$$(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{9}{16}\right)$$

You may continue your work for Q4 on this page.

5. (12 points) The vector field $\mathbf{F}$ is defined at all points in the xy–plane, besides the origin, as:

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

(a) **Without** first determining a potential: show that $\mathbf{F}$ is conservative on any simply–connected region in the plane that does not include the origin.

(b) Find a potential for $\mathbf{F}$ that applies on the upper–half–plane: $y > 0$. Ensure you provide a formula in x and y only and which is defined for all $y > 0$.

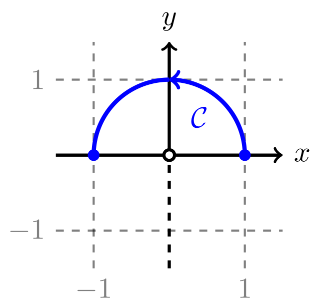
Hint: $\int \frac{1}{t^2+a^2}\, dt = \frac{1}{a} \arctan\left(\frac{t}{a}\right) + C$

(c) A potential can be defined on any simply–connected region of the xy–plane, excluding the origin, as:

$f(x,y) = \theta$ where θ is a continuous choice of polar angle for (x,y)

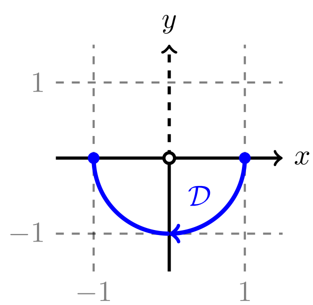
Use this to integrate $\mathbf{F}$ along the curve $\mathcal{C}$ sketched below. You do **not** have to justify that θ is a potential.

Hint: you could use the choice of polar angle $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ because it applies on the entire curve $\mathcal{C}$.



(d) Will the integral of $\mathbf{F}$ along the curve $\mathcal{D}$ below yield the same value obtained in part c? Justify your answer.

Hint: you could use the choice of polar angle $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$ because it applies on the entire curve $\mathcal{D}$.



Solution: (a) The curl of $\mathbf{F}$ is:

$$\left[\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right]\mathbf{k} = \left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right]\mathbf{k} = \mathbf{0}$$

Because the curl vanishes, $\mathbf{F}$ is conservative on any simply–connected region in the xy–plane.

(b) We calculate:

$$f(x,y) = \int \frac{\partial f}{\partial x}\, dx = \int -\frac{y}{x^2+y^2}\, dx = -\arctan\left(\frac{x}{y}\right) + C(y)$$

Next taking a y–derivative:

$$f_y(x,y) = \frac{x/y^2}{1+(x/y)^2} + C'(y) = \frac{x}{x^2+y^2} + C'(y)$$

and equating this to the second component of $\mathbf{F}$ gives:

$$\frac{x}{x^2+y^2} + C'(y) \overset{\text{set}}{=} \frac{x}{x^2+y^2}$$

and thus we may select $C(y) = 0$ to obtain potential:

$f(x,y) = -\arctan\left(\frac{x}{y}\right)$ which is defined for all $y > 0$

(c) We use the given potential $f(x,y) = \theta$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, which applies on the entire curve $\mathcal{C}$, to find:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}) = \pi - 0 = \pi$$

(d) We use the given potential $f(x,y) = \theta$ with $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$, which applies on the entire curve $\mathcal{D}$, to find:

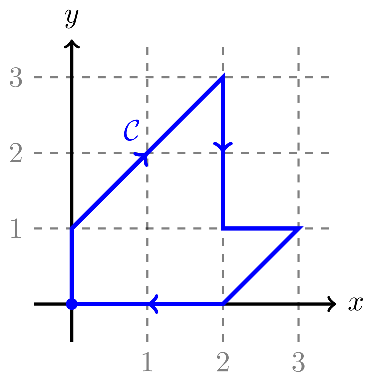
$$\int_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}) = \pi - 2\pi = -\pi$$

Hence: the values of found in part c and part d are different. This doesn't violate what we know about conservative vector fields, because $\mathbf{F}$ is not conservative on a region that contains both curves simulatneously.

You may continue your work for Q5 on this page.

6. (10 points) Let $\mathcal{C}$ be the oriented curve sketched below. Evaluate the line integral:

$$\oint_{\mathcal{C}} (3y - e^{\sin x} + 2)\,dx + (7x + \sqrt{y^4+1} - 3)\,dy$$



Solution: The vector field is defined and continuously differentiable on the entire xy–plane. The curve $\mathcal{C}$ is closed and oriented clockwise. So we may apply Green's theorem with a negative correction. Letting D denote the enclosed region, we find:

$$\oint_{\mathcal{C}} (3y - e^{\sin x} + 2)\,dx + (7x + \sqrt{y^4+1} - 3)\,dy = -\iint_D \frac{\partial}{\partial x}(7x + \sqrt{y^4+1} - 3) - \frac{\partial}{\partial y}(3y - e^{\sin x} + 2)\,dA = \cdots$$

$$\cdots = -\iint_D 4\,dA = -4 \cdot \text{area}(D) = -4 \cdot 4.5 = -18$$

You may continue your work for Q6 on this page.

7. (12 points) Let $\mathcal{S}$ be the portion of the unit sphere in the first octant: $x^2 + y^2 + z^2 = 1$ with $x, y, z \geq 0$. Orient $\mathcal{S}$ with upwards normals. Let $\mathbf{F}$ be a continuous vector field such that at points in $\mathcal{S}$:

   - $\mathbf{F}$ forms an angle of $\pi/6$ with the normal vector
   - $\mathbf{F}$ has magnitude $\|\mathbf{F}\| = y$

   Find the flux of $\mathbf{F}$ across $\mathcal{S}$.

   Hint: The dot product $\mathbf{F} \cdot d\mathbf{S}$ can be written in terms of lengths and angles. Since $\|d\mathbf{S}\| = dS$ you would obtain a scalar surface integral.

Solution: We compute:

$$\mathbf{F} \cdot d\mathbf{S} = \|\mathbf{F}\|\|d\mathbf{S}\| \cos\frac{\pi}{6} = y \cos\frac{\pi}{6}\ dS = \frac{\sqrt{3}}{2} y\ dS$$

On the unit sphere we have $y = \sin\phi \sin\theta$ and $dS = \sin\phi\ d\phi d\theta$. So the flux of $\mathbf{F}$ across the $\mathcal{S}$ is:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \frac{\sqrt{3}}{2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \sin\theta \cdot \sin\phi\ d\phi d\theta = \frac{\sqrt{3}}{2} \int_0^{\frac{\pi}{2}} \sin\theta\ d\theta \int_0^{\frac{\pi}{2}} \sin^2\phi\ d\phi = \cdots$$

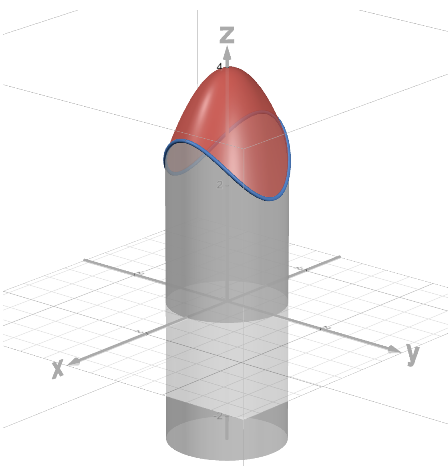
$$\cdots = \frac{\sqrt{3}}{2} \cdot 1 \cdot \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2}\cos(2\phi)\ d\phi = \frac{\sqrt{3}}{2} \cdot \frac{\pi}{4} = \frac{\sqrt{3}\pi}{8}$$

You may continue your work for Q7 on this page.

8. (12 points) The cylinder $x^2 + y^2 = 1$ intersects the paraboloid $z = 4 - x^2 - 2y^2$ along a curve $\mathcal{C}$. Let $\mathcal{S}$ be the part of the paraboloid lying above $\mathcal{C}$ and oriented with upwards normals. The surface is sketched in red below. Let

$\mathbf{F}(x, y, z) = \langle y, -2x, e^{x^2+y^2} - e\rangle$

Find: $\displaystyle\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$



Hint: use Stokes's Theorem to convert to a line integral. The last component of $\mathbf{F}$ simplifies nicely along the curve.

Solution: For Stokes's Theorem, due to upwards normals, we orient the intersection counterclockwise as viewed from above. We parametrize:

$\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, 4 - \cos^2\theta - 2\sin^2\theta\rangle$ with $0 \le \theta \le 2\pi$

We compute:

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin\theta, -2\cos\theta, e - e\rangle \cdot \langle -\sin\theta, \cos\theta, \star\rangle \; d\theta = -\int_0^{2\pi} \sin^2\theta + 2\cos^2\theta \; d\theta = \cdots$$

$$= -\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta + 1 + \cos 2\theta \; d\theta = -\int_0^{2\pi} \frac{3}{2} + \frac{1}{2}\cos 2\theta \; d\theta = -3\pi$$

You may continue your work for Q8 on this page.

9. (12 points) A continuously differentiable vector field $\mathbf{F}$ in xyz–space has divergence:

   div $\mathbf{F} = \nabla \cdot \mathbf{F} = r$

   where r is the standard cylindrical coordinate.

   Let $\mathcal{S}$ be the upper hemisphere:

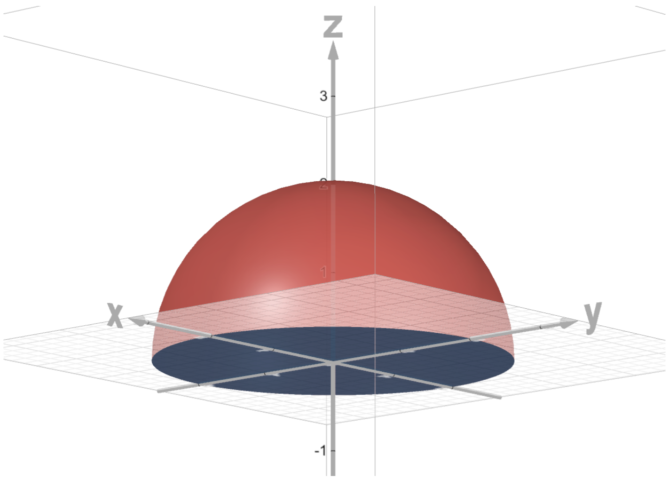
   $x^2 + y^2 + z^2 = 4$ with $z \geq 0$ and oriented with normals pointing up and sketched in red below

   and let $\mathcal{D}$ be the disk:

   $x^2 + y^2 \leq 4$ with $z = 0$ and oriented with normals pointing down and sketched in blue below.

   You are given that the flux of $\mathbf{F}$ across $\mathcal{S}$ is 6. Find the flux of $\mathbf{F}$ across $\mathcal{D}$.

   Hint: the lower hemisphere and the disk together form a closed surface. You could apply the divergence theorem to this closed surface as a step towards obtaining the solution. The only integral you would have to compute by hand is a triple integral. Spherical coordinates may be suitable.



Solution: The upper hemisphere with upwards normals, and the disk with downwards normals, together formed a closed surface with outwards normals. By the divergence theorem:

$$\iiint_{\text{enclosed}} \nabla \cdot \mathbf{F}\ dV = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S}$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho \sin\phi\ \rho^2 \sin\phi\ d\rho d\phi d\theta = 6 + \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S}$$

$$2\pi \cdot \frac{\pi}{4} \cdot 4 = 6 + \iint_{\text{disk}} \mathbf{F} \cdot d\mathbf{S}$$

$$2\pi^2 = 6 + \iint_{\text{disk}} \mathbf{F} \cdot d\mathbf{S}$$

So:

$$\iint_{\text{disk}} \mathbf{F} \cdot d\mathbf{S} = 2\pi^2 - 6$$

You may continue your work for Q9 on this page.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.