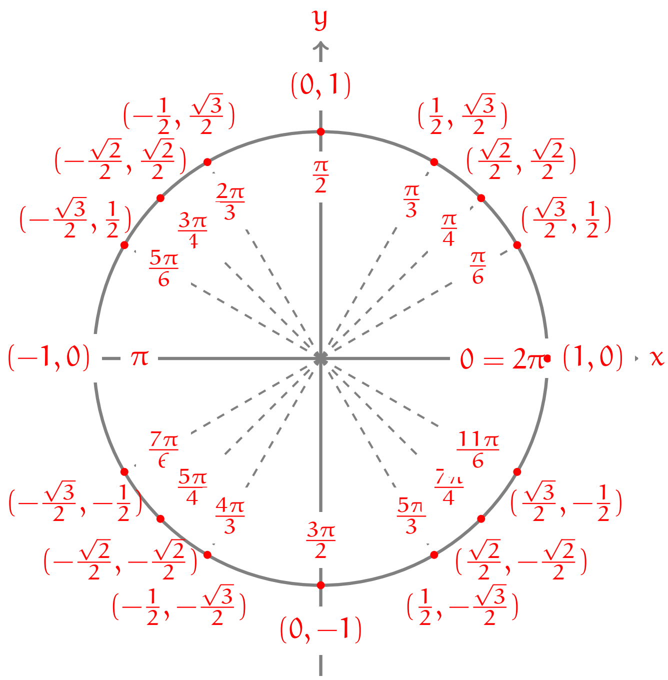
Math 226 – Reference Sheet. **Do not write on this sheet.**



Integral Formulas.

- $\int \frac{1}{1+t^2}\, dt = \arctan t + C$
- $\int \frac{1}{\sqrt{1-t^2}}\, dt = \arcsin t + C$
- $\int \frac{1}{t}\, dt = \ln|t| + C$

Trig Identities.

- power reduction: $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$
  $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$
- double angle: $\sin 2\theta = 2\sin\theta\cos\theta$
  $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Dot and Cross Products. Let $\vec{v}$, $\vec{w}$, $\vec{r}$ be vectors.

- $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$
- $\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta = [\text{parallelogram area}]$
- $|\vec{v}\cdot(\vec{w}\times\vec{r})| = [\text{parallelepiped volume}]$

Orthogonal Projections. Let $\vec{v}$ and $\vec{w}$ be vectors.

- $\text{proj}_{\vec{v}}(\vec{w}) = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$
- $\text{comp}_{\vec{v}}(\vec{w}) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|}$

Distance Formulas. Let Q be a point, $\mathcal{P}$ a plane with normal $\vec{n}$, and ℓ a line with direction vector $\vec{v}$.

- $\text{distance}(Q,\mathcal{P}) = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{\|\vec{n}\|}$ where P is a point on $\mathcal{P}$
- $\text{distance}(Q,\ell) = \frac{\|\overrightarrow{AB}\times\vec{v}\|}{\|\vec{v}\|}$ where P is a point on ℓ

Quadratic Surfaces. Standard equations.

- paraboloid: $z = x^2 + y^2$ 
- saddle: $z = x^2 - y^2$ 
- 1–sheeted hyperboloid: $x^2 + y^2 - z^2 = 1$ 
- 2–sheeted hyperboloid: $-x^2 - y^2 + z^2 = 1$ 
- ellipsoid—sphere if uniformly scaled:
  $x^2 + y^2 + z^2 = 1$ 
- double–cone: $z^2 = x^2 + y^2$ 

Tangent Planes. Let $f(x,y)$ be a function of two variables, $F(x,y,z)$ be a function of three variables, and C be a constant.

- tangent plane to the graph $z = f(x,y)$ at $P(a,b,\star)$ is:
  $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
- tangent plane to level set $F(x,y,z) = C$ at point P is:
  $\vec{\nabla}F(P)\cdot\vec{x} = \vec{\nabla}F(P)\cdot\vec{p}$

Directional Derivatives. Let f be a function and $\vec{u}$ be a unit direction in its domain.

- $D_{\vec{u}}f(P) = \vec{\nabla}f(P)\cdot\vec{u} = \|\vec{\nabla}f(P)\|\cos\theta$

Second Derivative Test.

- If P is a critical point of $f(x,y)$ and:
  $D = f_{xx}(P)f_{yy}(P) - (f_{xy}(P))^2$
  $T = f_{xx}(P) + f_{yy}(P)$
- then:
  $D < 0 \implies$ saddle
  $D > 0$ and $T > 0 \implies$ local minimizer
  $D > 0$ and $T < 0 \implies$ local maximizer

Lagrange Multipliers.

- An extremizer P of f subject to $g = C$ satisfies:
  $\vec{\nabla}f(P) = \lambda\vec{\nabla}g(P)$ or $\vec{\nabla}g(P) = \vec{0}$

Polar and Cylindrical Coordinates.

- $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$
- $x = r\cos\theta$ and $y = r\sin\theta$
- $dA = r\, dr d\theta$ and $dV = r\, dz dr d\theta$

Spherical Coordinates.

- $\rho^2 = x^2 + y^2 + z^2$ and $\tan\phi = \frac{r}{z}$
- $r = \rho\sin\phi$ and $z = \rho\cos\phi$
- $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$
- $dV = \rho^2\sin\phi\ d\rho d\phi d\theta$

Scalar Line Integrals.

- arclength $= \int_{\mathcal{C}} 1\ ds = \int_a^b \|\vec{r}'(t)\|\ dt$
- $\int_{\mathcal{C}} f\ ds$ where $ds = \|\vec{r}'(t)\|\ dt$

Vector Line Integrals.

- $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ where $d\vec{r} = \vec{r}'(t)\ dt$
- or $\int_{\mathcal{C}} P\ dx + Q\ dy + R\ dz$ where:
  $$\langle dx, dy, dz\rangle = \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt}\rangle\ dt$$

Fundamental Theorem of Line Integrals. Let $\vec{F}$ be conservative with potential f. Let $\vec{r}(t)$, $a \leqslant t \leqslant b$, be a path.

- If $\vec{r}(t)$, $a \leqslant t \leqslant b$, is a path in the domain of f, then:
  $$\int_{\vec{r}} \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Conservative Conditions.

- conservative vector fields are irrotational ($\text{curl} = \mathbf{0}$)
- irrot. + simply-conn. domain $\implies$ conservative

Curl. Let $\vec{F}$ be a vector field.

- $\text{curl}\,\vec{F} = \nabla\times\vec{F}$
- 2D vector fields $\vec{F} = \langle P, Q\rangle$ have: $\text{curl}\,\vec{F} = (Q_x - P_y)\vec{k}$

Parametrized Surfaces.

- parametrized surface $\vec{R}(u, v)$
  - infinitesimal normal: $d\vec{S} = \pm(\vec{R}_u\times\vec{R}_v)\ dudv$
  - infinitesimal surf. area: $dS = \|\vec{R}_u\times\vec{R}_v\|\ dudv$
- shortcut: parameters x and y
  - $d\vec{S} = \pm\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\rangle\ dxdy$
- shortcut: surface $z = f(r)$ with parameters r, θ
  - $d\vec{S} = \pm\langle -\frac{\partial z}{\partial r}\cdot r\cos\theta, -\frac{\partial z}{\partial r}\cdot r\sin\theta, r\rangle\ drd\theta$
  - $dS = r\sqrt{\left[\frac{\partial z}{\partial r}\right]^2 + 1}\ drd\theta$
- shortcut: sphere $\rho = R$ with parameters ϕ, θ
  - $d\vec{S} = (R\sin\phi)\langle x, y, z\rangle\ d\phi d\theta$
  - $dS = R^2\sin\phi\ d\phi d\theta$

Green's Theorem. Let D be a region in the xy–plane with closed boundary $\mathcal{C} = \partial D$, oriented so that D is on the left of the direction of $\mathcal{C}$.

- $\oint_{\mathcal{C}} P\ dx + Q\ dy = \iint_D Q_x - P_y\ dA$

Stokes's Theorem. Let $\mathcal{S}$ be an oriented surface with closed boundary $\mathcal{C} = \partial D$, oriented so that, when viewed from the direction to which the normals are pointing, $\mathcal{S}$ is on the left of the direction of $\mathcal{C}$.

- $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_{\mathcal{S}} \text{curl}\,\vec{F}\cdot d\vec{S}$