Math 226 Midterm C

Fa24

Mon Nov 25

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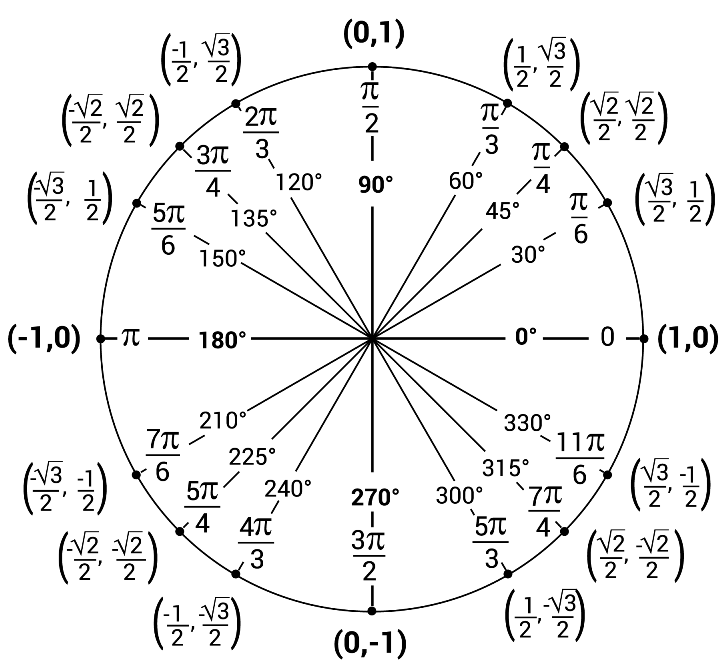
Instructions

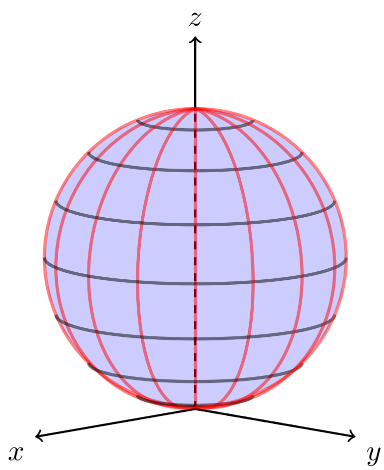
- This examination consists of 10 pages not including this cover page. Verify that your copy of this examination contains all 10 pages. If your examination is missing any pages then obtain a new copy immediately.
- You have 50 minutes to complete this examination.
- Do not use books, calculators, computers, tablets, or phones. You may use both sides of a single 3 in by 5 in index card but no other notes.
- Write legibly in the boxed area only. Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.
- If you run out of space: there are two pages at the end where you can continue your work.

Topic	C1	C2	C3	C4	C5
Max Points	4	4	4	4	4



1. (C1) (4 points) This question has multiple parts.

(a) Let B be the shifted–up solid ball given by $x^2 + y^2 + (z-2)^2 \leq 4$. Find the bounds in order $d\rho d\phi d\theta$ if B is the region of integration.



$\leq \rho \leq$
$\leq \phi \leq$
$\leq \theta \leq$

Solution: We rewrite the inequalities as:

$$x^2 + y^2 + z^2 - 4z + 4 \leq 4 \implies x^2 + y^2 + z^2 \leq 4z \implies \rho^2 \leq 4\rho\cos\phi \implies \rho \leq 4\cos\phi$$

So the bounds on ρ are $0 \leq \rho \leq 4\cos\phi$.

Using the picture we have $0 \leq \phi \leq \frac{\pi}{2}$ (the ball is above the xy-plane) and $0 \leq \theta \leq 2\pi$.

(b) Find the integral by converting to spherical coordinates.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 3e^{(x^2+y^2+z^2)^{3/2}}\ dzdydx$$

Solution: The region of integration is the part of $x^2 + y^2 + z^2 = 1$ in the first octant. The given integral thus equals:

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 3e^{\rho^3} \cdot \rho^2 \sin\phi\ d\rho d\phi d\theta = \int_0^{\pi/2} \int_0^{\pi/2} (e-1)\sin\phi\ d\phi d\theta = \int_0^{\pi/2} (e-1)\ d\theta = \frac{(e-1)\pi}{2}$$

2. (C2) For this problem consider the curve $\mathcal{C}$ given by the intersection of the cylinder $y^2 + z^2 = 9$ and the surface $x = yz$, but only the half of this curve with $z \geq 0$.

(a) Parametrize $\mathcal{C}$. Hint: use the fact that $z \geq 0$ to decide the limits on your parameter.

Solution: The part of the circle $y^2 + z^2 = 9$ in the yz-plane with $z \geq 0$ can be described with $y = 3\cos\theta$ and $z = 3\sin\theta$ with $0 \leq \theta \leq \pi$. Using that $x = yz = (3\cos\theta)(3\sin\theta)$ we get parametrization for the curve:

$$\mathbf{r}(\theta) = \langle 9\cos\theta\sin\theta, 3\cos\theta, 3\sin\theta\rangle \text{ with } 0 \leq \theta \leq \pi$$

(b) Compute the vector line integral:

$$\int_{\mathcal{C}} 1\ dx + y\ dy + z\ dz$$

Hint: simplify using some of: $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ or $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ or $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

Solution: We first rewrite our parametrization using the hint as:

$$\mathbf{r}(\theta) = \left\langle \frac{9}{2}\sin 2\theta, 3\cos\theta, 3\sin\theta \right\rangle \text{ with } 0 \leq \theta \leq \pi$$

and then

$$\mathbf{r}'(\theta) = \langle 9\cos 2\theta, -3\sin\theta, 3\cos\theta\rangle$$

So:

$$\int_{\mathcal{C}} 1\ dx + y\ dy + z\ dz = \int_0^{\pi} 9\cos(2\theta) - 9\sin\theta\cos\theta + 9\sin\theta\cos\theta\ d\theta = \int_0^{\pi} 9\cos(2\theta)\ d\theta = \frac{9\sin(\pi) - 9\sin(0)}{2} = 0$$

(c) Set up but **do not compute** a single variable integral that will equal the arclength of $\mathcal{C}$.
Note: the single variable involved should be your parameter.

Solution: It is:

$$\int_{\mathcal{C}} 1\ ds = \int_0^{\pi} \|\mathbf{r}'(\theta)\|\ d\theta = \int_0^{\pi} \sqrt{81\cos^2(2\theta) + 9\sin^2\theta + 9\cos^2\theta}\ d\theta = \int_0^{\pi} 3\sqrt{9\cos^2(2\theta) + 1}\ d\theta$$

3. (C3) This problem has multiple parts.

(a) Consider the vector field:

$$\mathbf{F}(x,y,z) = (2xyz)\,\mathbf{i} + \left(x^2z + e^{z^2}\right)\mathbf{j} + \left(x^2y + 2yze^{z^2}\right)\mathbf{k}$$

(i) Show that the vector field is conservative by finding a potential function.

Solution: Using guess–and–check I found potential:

$$f(x,y,z) = x^2yz + ye^{z^2}$$

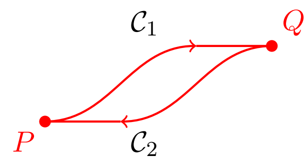
(ii) Consider the path $\mathbf{r}(t) = \langle t+1, t^2-1, e^t\rangle$ with $0 \le t \le 1$ and find:

$$\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{s}$$

Solution: We use the potential function to find:

$$\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{s} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) = f(2,0,e) - f(1,-1,1) = 0 - (-1-e) = 1+e$$

(b) Consider the oriented curves $\mathcal{C}_1$ and $\mathcal{C}_2$ depicted below.



Suppose that $\mathbf{G}(x,y)$ is a "nice" vector field that is defined and **<u>conservative</u>** everywhere and that:

$$\int_{\mathcal{C}_1} \mathbf{G}\cdot d\mathbf{s} = 4$$

For each part: decide whether there is enough information to determine the value. If there IS enough information, then compute the value. If there IS NOT enough information, then write "undetermined."

(i) $\operatorname{curl}\mathbf{G}(P) =$

(ii) $\displaystyle\int_{\mathcal{C}_2} \mathbf{G}\cdot d\mathbf{s} =$

Solution: (i) Because $\mathbf{G}$ is conservative it follows that $\operatorname{curl}\mathbf{G}(P) = \mathbf{0}$.

(ii) The paths $\mathcal{C}_1$ and $\mathcal{C}_2$ have opposite starts and ends. Because $\mathbf{G}$ is conservative it follows that:

$$\int_{\mathcal{C}_2} \mathbf{G} \cdot d\mathbf{s} = -\int_{\mathcal{C}_1} \mathbf{G} \cdot d\mathbf{s} = -4$$

4. (C4) This problem has multiple parts.

(a) Consider the surface $\mathcal{W}$ (called the Whitney umbrella) parametrized by $\mathbf{X}(u, v) = \langle uv, u, v^2 \rangle$.

(i) Set up **but do not compute** a double integral in u and v that will equal the **surface area** of the portion of $\mathcal{W}$ with $0 \leq u \leq 2$ and $0 \leq v \leq 2$. Note: while you do not have to compute the integral, the integrand (expression to integrate) should be computed and simplified.

Solution: We compute:

$$\mathbf{X}_u \times \mathbf{X}_v = \langle v, 1, 0 \rangle \times \langle u, 0, 2v \rangle = \langle 2v, -2v^2, -u \rangle$$

and so:

$$\iint_{\mathbf{X}} 1 \, dS = \int_0^2 \int_0^2 \|\mathbf{X}_u \times \mathbf{X}_v\| \, dudv = \int_0^2 \int_0^2 \sqrt{4v^2 + 4v^4 + u^2} \, dudv$$

(ii) Find an equation of the tangent plane to the Whitney umbrella $\mathcal{W}$ at the point $(2, 1, 4)$.
Note: your final answer should be an equation of a plane!

Solution: First we set up $(2, 1, 4) = (uv, u, v^2) \implies (u, v) = (1, 2)$.
Next: a normal to the surface at $(2, 1, 4)$ can be found with the help of previous work:

$$\mathbf{X}_u(1, 2) \times \mathbf{X}_v(1, 2) = \langle 2v, -2v^2, -u \rangle|_{(u=1, v=2)} = \langle 4, -8, -1 \rangle$$

An equation of the plane is thus:

$$4(x - 2) - 8(y - 1) - 1(z - 4) = 0$$

(b) Let $\mathcal{S}$ be the portion of the paraboloid $z = 5 - x^2 - y^2$ with $z \geq 1$ and oriented with **upwards normals**. Find:

$$\iint_{\mathcal{S}} \left\langle x, y, x^2 + y^2 \right\rangle \cdot d\mathbf{S}$$

Note: For a graph $z = f(r)$ it will turn out that $\mathbf{X}_r \times \mathbf{X}_\theta = \langle -f'(r) r \cos\theta, -f'(r) r \sin\theta, r \rangle$. You may use this.

Solution: Note that in cylindrical coordinates, the surface is $z = 5 - r^2$ in which case $z \geq 1$ occurs when $5 - r^2 \geq 1 \implies 4 \geq r^2 \implies 2 \geq r$. We parametrize the surface using:

$$\mathbf{X}(r, \theta) = \langle r\cos\theta, r\sin\theta, 5 - r^2 \rangle \text{ with } 0 \leq r \leq 2 \text{ and } 0 \leq \theta \leq 2\pi$$

Then:

$$\mathbf{X}_r \times \mathbf{X}_\theta = \langle \cos\theta, \sin\theta, -2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

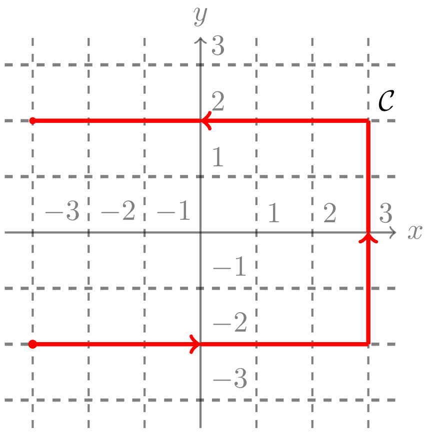
which are upwards as desired. So:

$$\iint_{\mathcal{S}} \left\langle x, y, x^2 + y^2 \right\rangle \cdot d\mathbf{S} = \int_0^2 \int_0^{2\pi} \langle r\cos\theta, r\sin\theta, r^2 \rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle \, d\theta dr = \cdots$$

$$\cdots = \int_0^2 \int_0^{2\pi} 3r^3 \; d\theta dr = 2\pi \cdot \frac{3}{4} \cdot 2^4 = 24\pi$$

5. (C5) This problem has multiple parts.

(a) Let $\mathcal{C}$ be the oriented curve depicted below and find the given integral. Hint: a direct approach is not recommended. Use Green's Theorem by closing off the curve.



$$\int_{\mathcal{C}} \left[4y + \sin\left(e^{e^x}\right)\right] dx + [6x]\, dy$$

Solution: Let $\mathcal{C}'$ be the line segment from $(-3, 2)$ to $(-3, -2)$. Then $\mathcal{C} + \mathcal{C}'$ is counterclockwise closed and so we can compute an integral of the given vector field over this curve using Green's theorem:

$$\int_{\mathcal{C}+\mathcal{C}'} \left[4y + \sin\left(e^{e^x}\right)\right] dx + \Big[6x\Big]\, dy = \iint_{\text{enclosed}} (6-4)\ dA = 2 \cdot \text{area(enclosed)} = 48$$

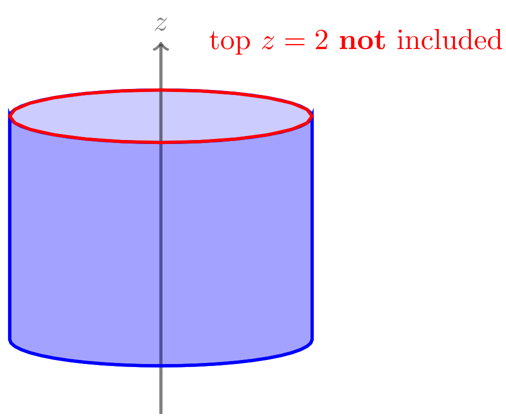
Next we can compute the integral of $-\mathcal{C}'$ using parametrization $\mathbf{r}(y) = \langle -3, y\rangle \implies \mathbf{r}'(y) = \langle 0, 1\rangle$ with $-2 \le y \le 2$ as:

$$\int_{-\mathcal{C}'} \left[4y + \sin\left(e^{e^x}\right)\right] dx + [6x]\, dy = \int_{-2}^{2} -18\ dy = -72$$

So:

$$\int_{\mathcal{C}} = \int_{\mathcal{C}+\mathcal{C}'} + \int_{-\mathcal{C}'} = 48 - 72 = -24$$

(b) Let $\mathcal{S}$ be the surface consisting of the portion of the cylinder $x^2 + y^2 = 4$ between $-2 \le z \le 2$ along with its bottom at $z = -2$, but **not** its top. Orient $\mathcal{S}$ with **inward** normals. Find the given integral. Hint: Stokes.



$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \ \text{ if } \ \mathbf{F} = -yz\mathbf{i} + xz\mathbf{j} + e^{x^2y^2z^2}\,\mathbf{k}$$

Solution: The boundary ∂S of $\mathcal{S}$ is its top edge : $x^2 + y^2 = 4$ with $z = 2$. To be oriented compatibly with the given orientation, ∂S should be counterclockwise oriented (when viewed from above). We parametrize it as $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 2\rangle \implies \mathbf{r}'(t) = \langle -2\cos t, 2\cos t, 0\rangle$ with $0 \le t \le 2\pi$. Then by Stokes's Theorem:

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \langle -4\sin t, 4\cos t, \star\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle \; d\theta = \int_0^{2\pi} 8 \; d\theta = 16\pi$$

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