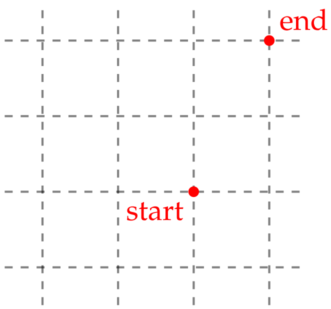
A. **Vectors.** Vectors capture displacements, which are at the core of calculus, particularly when we allow the displacements to become infinitesimal. The change in position from a startpoint to an endpoint is referred to as a **displacement** and is captured by a **vector**. They are determined by the changes (Δ's) in coordinates form **start** to **end**, and we call those changes the **components** of the vector.

If you are born and die in the same place, your life displacement vector is zero.

sketch | **component notation**

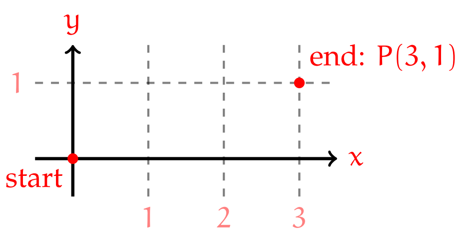


Precisely, the first component is the change in x from start to end, and the second is the change in y, and the third (excuse me?) is the change in z. Vectors are typically denoted either with **boldface** like $\mathbf{v}$ or with an arrow (or half-arrow) on top, like $\vec{v}$.

Vectors remain unchanged under **translation**, because only the changes in coordinates from start to end matter.

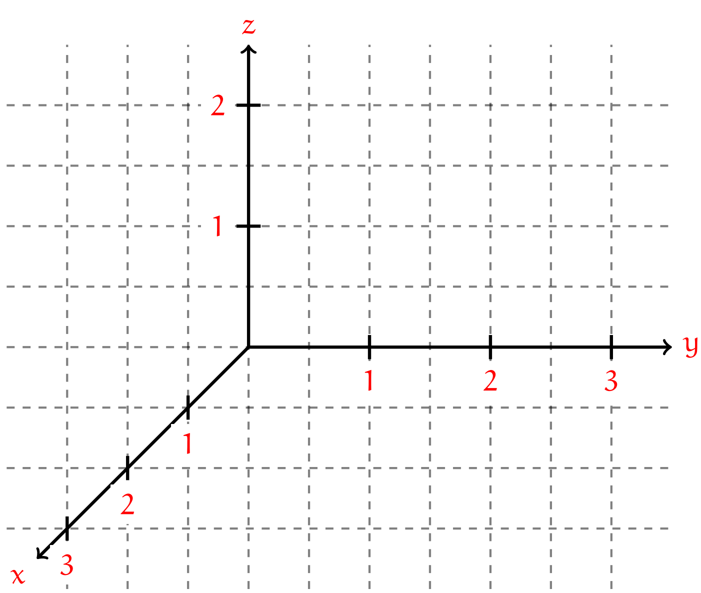
Translation means you move the vector without rotating or stretching, though you should make rotating and stretching a part of your daily routine.

Of special interest is the displacement vector from the origin to a given point P, which we call the **position vector** of P, and write as $\vec{p}$.



We can also look at vectors similarly in three dimensions. For example let us examine the position vector of $P = (1, 3, 2)$.

There are many correct ways to represent a 3D axis on a 2D grid. In this picture, the x–axis goes "towards you", the y–axis goes to the right, and the z–axis goes up. To preserve some notion of correct distance given our perspective, one unit in x is the length of the diagonal of a grid square, while one unit in y or z is the length of two sides of a grid square.



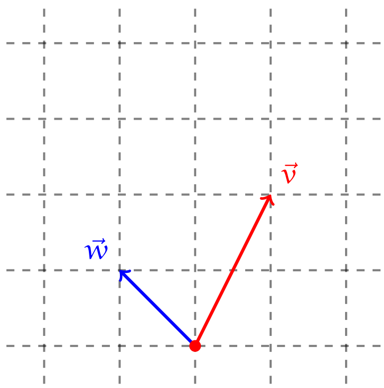
B. **Basic Vector Operations.** We can do some algebra with vectors.

Do we want to do algebra with vectors? Whatever way we define algebra, it should have some reasonable geometric representation, otherwise it is pointless. The operations we define here are easy to understand geometrically. Things like multiplying two vectors is trickier, and will be addressed later.

Vector Addition: Vectors can be added component-by-component, provided they have the same number of components.

$$\underbrace{\langle 1, 2\rangle}_{\vec{v}} + \underbrace{\langle -1, 1\rangle}_{\vec{w}} =$$

Vector addition can be executed **geometrically** too. First translate the second vector so its **tail** (start) coincides with the **tip** (end) of the first vector. Then the **resultant** vector is drawn from the tail of the first vector to the tip of the second. This resultant vector is the sum.



Another method is the **parallelogram** method, which entails translating the vectors to have the **same start**, creating a parallelogram with these vectors as sides, in which case the diagonal of the paralleogram, having the same start as the two vectors, is their sum.

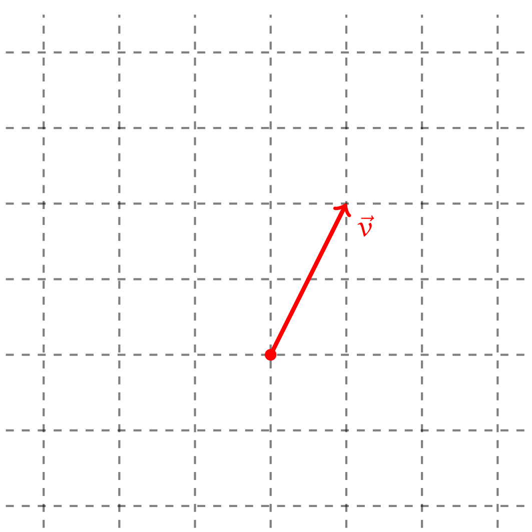
Scalar Multiplication: Vectors can also be **multiplied** by **scalars** (real numbers):

$$2\langle 1, 2\rangle =$$

$$-1\langle 1, 2\rangle =$$

Why would we call real numbers scalars? Because they are meant for scaling the size of vectors! Don't question further.

and this can be executed geometrically as depicted.



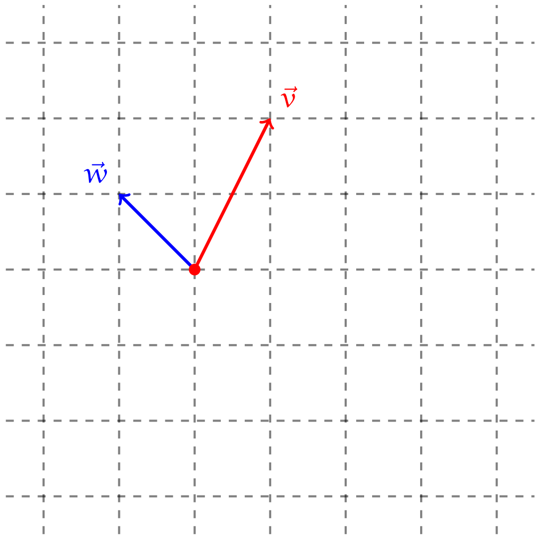
A scalar c bigger than 1 stretches the vector by a factor of c. A scalar between 0 and 1 shortens the vector. A negative scalar reverses the direction of the vector.

Example 1. Use the provided $\vec{v}$ and $\vec{w}$ to sketch:

$2(0.5\vec{v} - 1\vec{w}) =$

$0\vec{v} =$

The **zero vector** is the vector $\mathbf{0} = \langle 0, \ldots, 0 \rangle$.



In completing this excercise we are applying some rules of vector algebra:
distributivity: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$
associativity: $c_1(c_2\vec{v}) = (c_1c_2)\vec{v}$
For your hungry mathematical mind, think of distributivity as being allowed to **distribute** (surprise) multiplication over addition, and associativity as being allowed to move parentheses around with abandon.

C. **Dot Products and Length.** We have not talked about multiplying vectors yet. There are two useful ways to do so. We discuss the first of these: the **dot product**.

We have not talked about it yet because the right multiplication might not be what you think the right multiplication is.

If $\vec{v}$ and $\vec{w}$ have the same number of components then their **dot product** is:

$$\underbrace{\langle v_1, v_2, \ldots, v_n \rangle}_{\vec{v}} \cdot \underbrace{\langle w_1, w_2, \ldots, w_n \rangle}_{\vec{w}} =$$

If they do not have the same number of components, the dot product ain't defined.

$$\langle 1, 2, 3 \rangle \cdot \langle 3, 2, 1 \rangle =$$

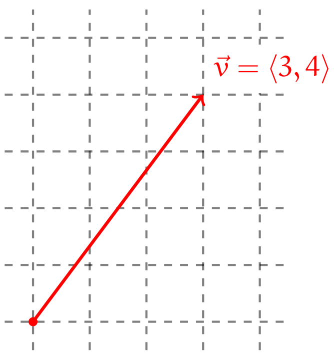
$$\langle 3, 4 \rangle \cdot \langle 3, 4 \rangle =$$

Note that (vector) · (vector) = (scalar) where you would not dare have forgotten that a scalar is a real number.

What is the meaning of the scalar we get out?

Does it even have a meaning?

We first investigate the case of a vector being **dotted** with itself.



The **length** of a vector is:

$\|\vec{v}\| =$

Note that the length is always **nonnegative**.

And thus $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Example 2. Find the length of the vector $\vec{u} = -\frac{1}{13}\langle 3, 4, 12 \rangle$.

The property we use is that $\|c\vec{v}\| = |c|\,\|\vec{v}\|$, grounded in the intuitive idea that scaling a vector by a factor scales its length by the same factor, or rather the absolute value of that factor, because lengths are never negative!

A **unit vector** is a vector with length 1.