Example 1. Let $\vec{v}$ and $\vec{w}$ be **unit** vectors so that $\vec{v} \cdot \vec{w} = \frac{1}{2}$ and find:

$\|2\vec{v} + \vec{w}\|^2 =$

Remember a unit vector is a vector with length 1.

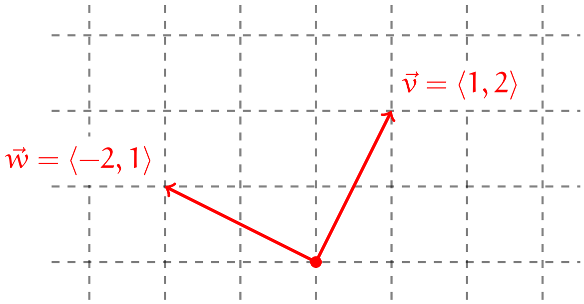
Also remember: $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$

The new dot product properties:
distributivity: $\vec{v} \cdot (\vec{w} + \vec{r}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{r}$
and: $(\vec{v} + \vec{w}) \cdot \vec{r} = \vec{v} \cdot \vec{r} + \vec{w} \cdot \vec{r}$
commutativity: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
and: $c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w})$

Distributivity is all about distributing multiplication over addition. Commutativity tells us we can change the order, in this case of multiplication, without changing the result. It's not like we'll ever encounter any multiplication that is not commutative. Right...right professor?

A. **Dot Products and Angles.** The dot product of two vectors is a **scalar**. What if that scalar were **zero**?

Would that be scary?



Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** when:

$$\vec{v} \cdot \vec{w} =$$

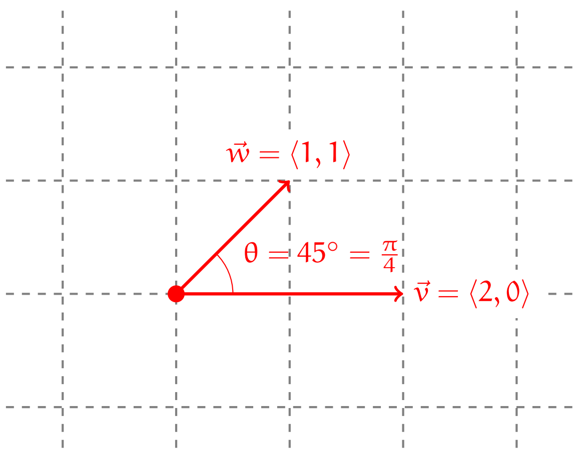
in which case we write $\vec{v} \perp \vec{w}$.

Orthogonal and perpendicular mean the same thing. Tomato tomato.

What if the scalar we get from the dot product of $\vec{v}$ and $\vec{w}$ is **nonzero**. What does this tell us about the vectors $\vec{v}$ and $\vec{w}$?

Consider two vectors and the angle between them.

The angle could be the short way around as depicted, or the long way around, but it don't matter. What follows is still true.



For $\vec{v} = \langle 1, 1 \rangle$ and $\vec{w} = \langle 2, 0 \rangle$ let us compute:

$$\vec{v} \cdot \vec{w} =$$

$$\|\vec{v}\|\|\vec{w}\| \cos \theta =$$

If θ is the angle between $\vec{v}$ and $\vec{w}$ then:

$$\vec{v} \cdot \vec{w} =$$

which agrees nicely with the fact that $\vec{v} \cdot \vec{w} = 0$ when $\theta =$

We have not even attempted to really establish why this is always true. If you are curious why, ask your TA or professor in office hours. We can at least cover a basic case: when one of the vectors is parallel to the x-axis.

Example 2. Find all vectors orthogonal to:

$\vec{v} = \langle 3, 4 \rangle$

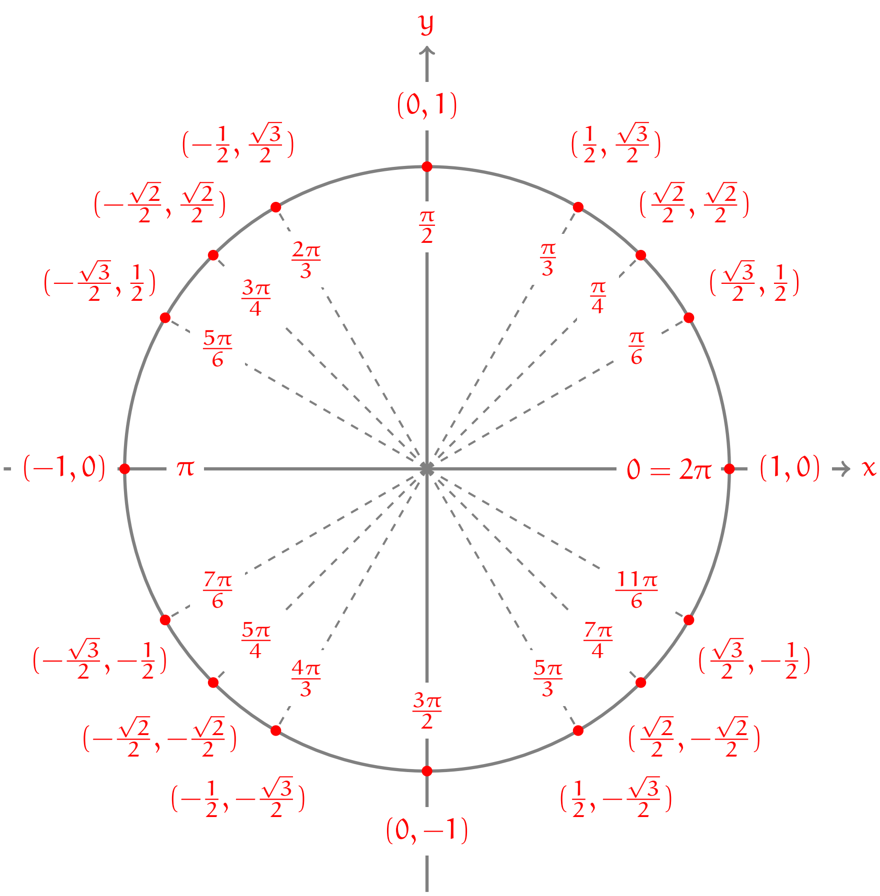
that have the same length as $\vec{v}$.

The rotation of vector $\langle a, b \rangle$ counterclockwise by $90°$ is $\langle -b, a \rangle$ while its rotation clockwise by $90°$ is $\langle b, -a \rangle$

Example 3. Find the angle between the vectors:

$\vec{v} = \langle 1, 0, 1 \rangle$ and $\vec{w} = \langle -1, 1, 0 \rangle$

You may find the **unit circle** helpful for locating the correct value.

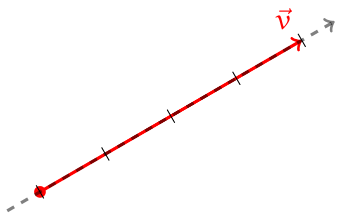


If θ tells you the angle of counterclockwise rotation from the positive x-axis to the position vector of the point, then remember that $\cos\theta$ is the x-coordinate of the point and $\sin\theta$ is the y-coordinate.

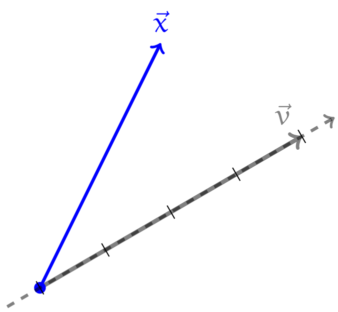
The unit circle also helps us see that if $\vec{v} \cdot \vec{w}$ is negative, then the angle between the vectors is larger that $90°$. Or more precisely, a counterclockwise rotation of one vector to another would have to be by an angle of magnitude greater than $90°$.

B. **Projections.** A **nonzero** vector $\vec{v}$ can be **normalized** to give a **unit direction**:

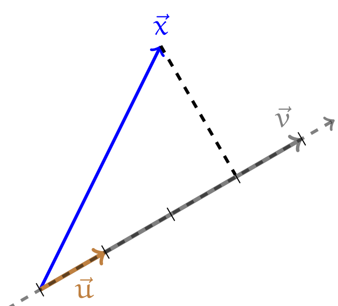
$\vec{u} =$



Given **another** vector $\vec{x}$, translated so it and $\vec{v}$ share the same start, we can construct a **right triangle** with hypotenuse $\vec{x}$ and one side parallel to $\vec{v}$. We can treat the side parallel to $\vec{v}$ as a vector itself, that we call the **orthogonal projection** of $\vec{x}$ onto $\vec{v}$ and denote by $\text{proj}_{\vec{v}}(\vec{x})$.



Then $\text{proj}_{\vec{v}}(\vec{x}) = c\vec{u}$ for a constant c called the **scalar component** of $\vec{x}$ onto $\vec{v}$ and denoted $c = \text{comp}_{\vec{v}}(\vec{x})$.



Orthogonal Projection and Scalar Component.

If we **normalize** $\vec{v}$ to be a unit vector:

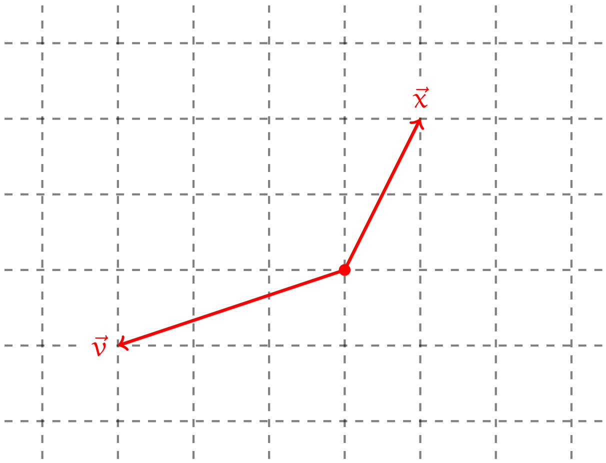
$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

Then we have the following formulas:

$\text{comp}_{\vec{v}}(\vec{x}) =$

$\text{proj}_{\vec{v}}(\vec{x}) =$

Example 4. Find the orthogonal projection of $\vec{x} = \langle 1, 2 \rangle$ onto $\vec{v} = \langle -3, -1 \rangle$ and find the scalar component of $\vec{x}$ along $\vec{v}$.



What is the orthogonal projection of me onto my neighbor?