A. **Cross Products.** Let us talk about a second useful way to multiply vectors. We call it the cross product.

$$(\text{3D vector}) \times (\text{3D vector}) = (\text{3D vector})$$

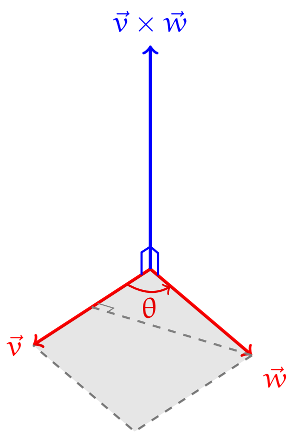
Let us explain what characterizes it.

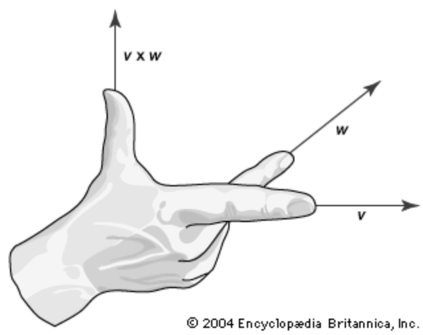
Cross products are only for 3D vectors? I wonder why?

By "smaller angle" we mean an angle in the range $0 \leqslant \theta \leqslant \pi$.

If $\vec{v}$ and $\vec{w}$ are 3D vectors with smaller angle θ between them, then their cross product $\vec{v} \times \vec{w}$ is the 3D vector that:

- is orthogonal to both $\vec{v}$ and $\vec{w}$
- has direction determined by the **righthand rule**





In words, to execute the righthand rule, simultaneously **extend** (don't curl) your index finger in the direction the **first vector** is pointing, and **curl** your middle finger in the direction the **second vector** is pointing, in which case your thumb points in the direction of the cross product.

- has length equal to the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$, i.e.:

$$\|\vec{v} \times \vec{w}\| =$$

For any 3D vectors $\vec{v}$ and $\vec{w}$ we have the following properties.

Anti–Commutativity: $\vec{w} \times \vec{v} =$

Self–Annihilating: $\vec{v} \times \vec{v} =$

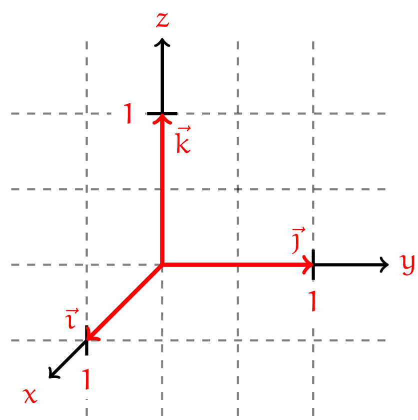
Oh dear lord I cannot just change the order of multiplication like I have been doing my entire dang life?

Example 1. Compute the cross products involving the special vectors:

$\vec{\imath} =$

$\vec{\jmath} =$

$\vec{k} =$



$\vec{\imath}, \vec{\jmath}, \vec{k}$ is really physics notation. Mathematicians might prefer $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

$\vec{\imath} \times \vec{\jmath} =$

$\vec{\imath} \times \vec{k} =$

$\vec{\jmath} \times \vec{k} =$

$\vec{\imath} \times (\vec{\imath} \times \vec{\jmath}) =$

$(\vec{\imath} \times \vec{\imath}) \times \vec{\jmath} =$

Oh my god. You **cannot** freely move parentheses around? That's SO messed up. This is referred to as the **failure** of associativity: $(\vec{v} \times \vec{w}) \times \vec{r} \neq \vec{v} \times (\vec{w} \times \vec{r})$. Associativity is all about moving parentheses around.

Next use the idea that any 3D vec can be written in terms of these special vecs:

$$\boxed{\langle a, b, c \rangle =}$$

along with the new properties in the margin to find:

$\langle 1, 2, 0 \rangle \times \langle 2, 0, 0 \rangle$

distributivity:
$(\vec{v} + \vec{w}) \times \vec{r} = \vec{v} \times \vec{r} + \vec{w} \times \vec{r}$
$\vec{v} \times (\vec{w} + \vec{r}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{r}$
commutativity with scalars:
$(c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) = c(\vec{v} \times \vec{w})$

B. **Computing Cross Products.** So far cross-products seem tough to compute.

Is there not a magic formula?

The cross–product $\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle$ equals the **determinant**:

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which means it equals:

$$+\begin{vmatrix} \vec{\imath} & & \\ & a_2 & a_3 \\ & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} & \vec{\jmath} & \\ a_1 & & a_3 \\ b_1 & & b_3 \end{vmatrix} + \begin{vmatrix} & & \vec{k} \\ a_1 & a_2 & \\ b_1 & b_2 & \end{vmatrix}$$

We call a rectangular array of entries a matrix. The determinant of a 2 (rows) by 2 (columns) matrix is written with notation that looks like the absolute value of the matrix, though we are not taking an absolute value, but are computing:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and this is what we are computing thrice as part of calculating the cross product.

Example 2. Find a nonzero vector orthogonal to $\vec{v} = \langle 4, 5, 6\rangle$ and $\vec{w} = \langle 7, 8, 10\rangle$.

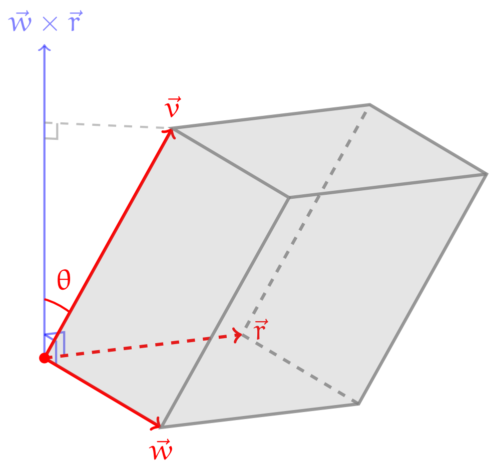
From now on, if you ever need a vector orthogonal to two other 3D vectors, cross products better leap into your mind!

C. **Scalar Triple Product.** The cross product and dot product do not have to operate in isolation. We can execute them in succession:

> The **scalar triple product** of 3D vectors $\vec{v}$, $\vec{w}$, $\vec{r}$ is:
>
> $\vec{v} \cdot (\vec{w} \times \vec{r})$

It has an important geometric meaning. Each pair of vectors from $\vec{v}$, $\vec{w}$, $\vec{r}$ forms a parallelogram, and together they form an object called a **parallelepiped**.

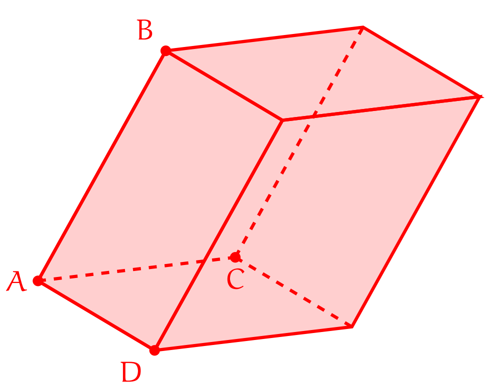
Say that three times fast. You can think of it as a slanted cube.



> The **volume** of the parallelepiped formed by $\vec{v}$, $\vec{w}$, $\vec{r}$ equals:
>
> or in other words equals the **absolute value** of their **scalar triple product**.

We use the absolute value because, for us, the word **volume** should correspond to a nonnegative quantity.

Example 3. Find the volume of the parallelepiped with vertex $A(1, 0, 3)$ adjacent to vertices $B(2, 2, 6)$, $C(5, 5, 9)$, $D(8, 8, 13)$.



This is probably not even close to how this parallelepiped actually looks in xyz-space. Nonetheless we make a sketch because it helps organize our thoughts. And god knows I need help with that.