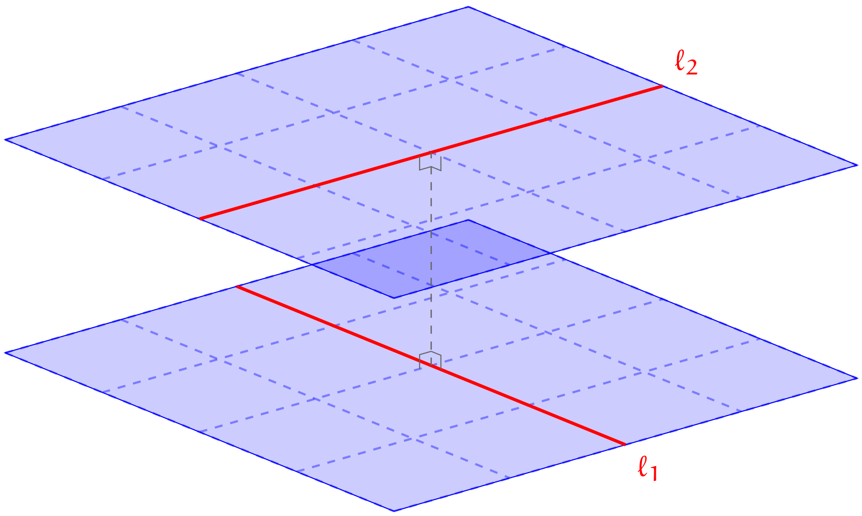
Example 1. Two lines are **skew** if they are not parallel and not intersecting.

When we say that lines are not parallel, we mean that their direction vectors are not parallel, which means their direction vectors are not scalar multiples of each other.



Consider the lines with parametrizations:

$$\ell_1 : \vec{r}_1(t) = \langle 1 + t, -2 + 3t, 4 - t \rangle$$

$$\ell_2 : \vec{r}_2(t) = \langle 2t, 3 + t, -1 + 4t \rangle$$

(a) Assess whether the lines are parallel to each other.

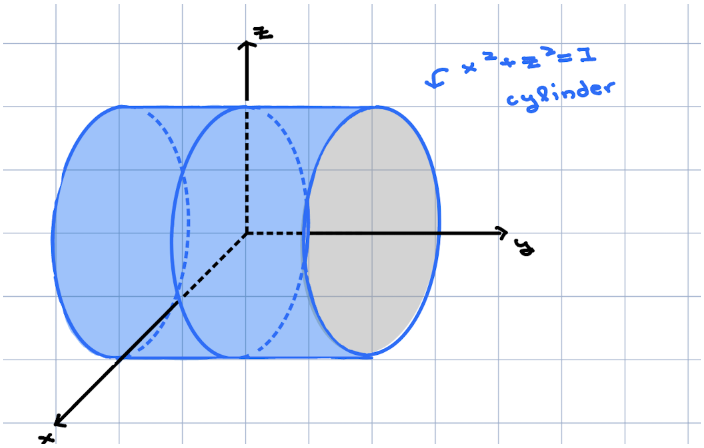
(b) Find the distance between the two lines.

Any two lines can be situated in parallel planes, as we can see by using a cross product to find a vector that is normal to both lines. When the lines are not parallel to each other, the distance between those planes is the same as the distance between the lines.

A. **Cylinders and Slices.** Planes are examples of **surfaces**, specifically they are surfaces defined by an equation involving a linear function of two variables. A **quadric surface** is defined by an equation involving a degree 2 polynomial of two variables.

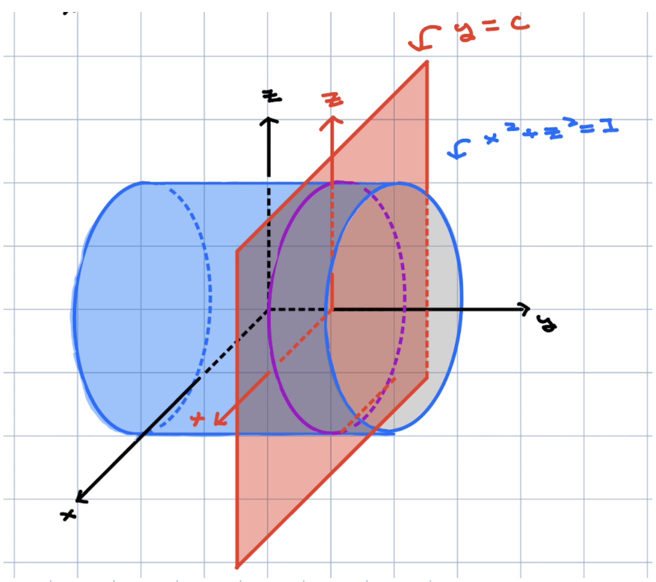
A **polynomial** is a sum of terms, each of which looks like a scalar times some variables, like $2x + 3xy + 5x^2y$. The **degree** is related to the maximum number of variables that are multiplied together. For example the above polynomial has degree 3 because of the term $5x^2y = 5xxy$ which has 3 variables multiplied together.

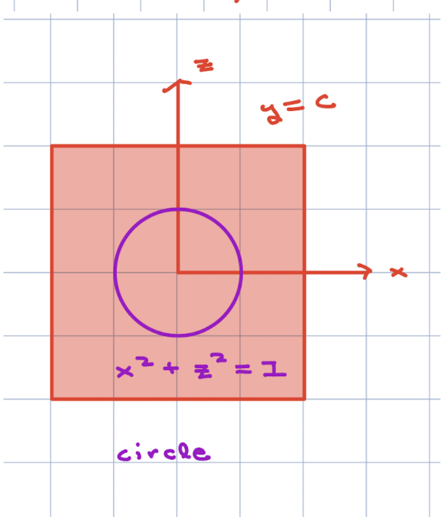
For example the equation $x^2+z^2 = 1$ defines the circular cylinder, also in Desmos.



Remember, an equation defines a surface by giving you a rule to assess whether or not a point (x, y, z) is on the surface. In this case the rule is $x^2 + z^2 = 1$. Hence $(1, 2, 0)$ is on the surface, because $1^2 + 0^2 = 1$, but $(1, 2, 1)$ is not on the surface, because $1^2 + 1^2 \neq 1$.

We could reconstruct its graph by looking at its $y = c$ **slices**, where c is a constant.

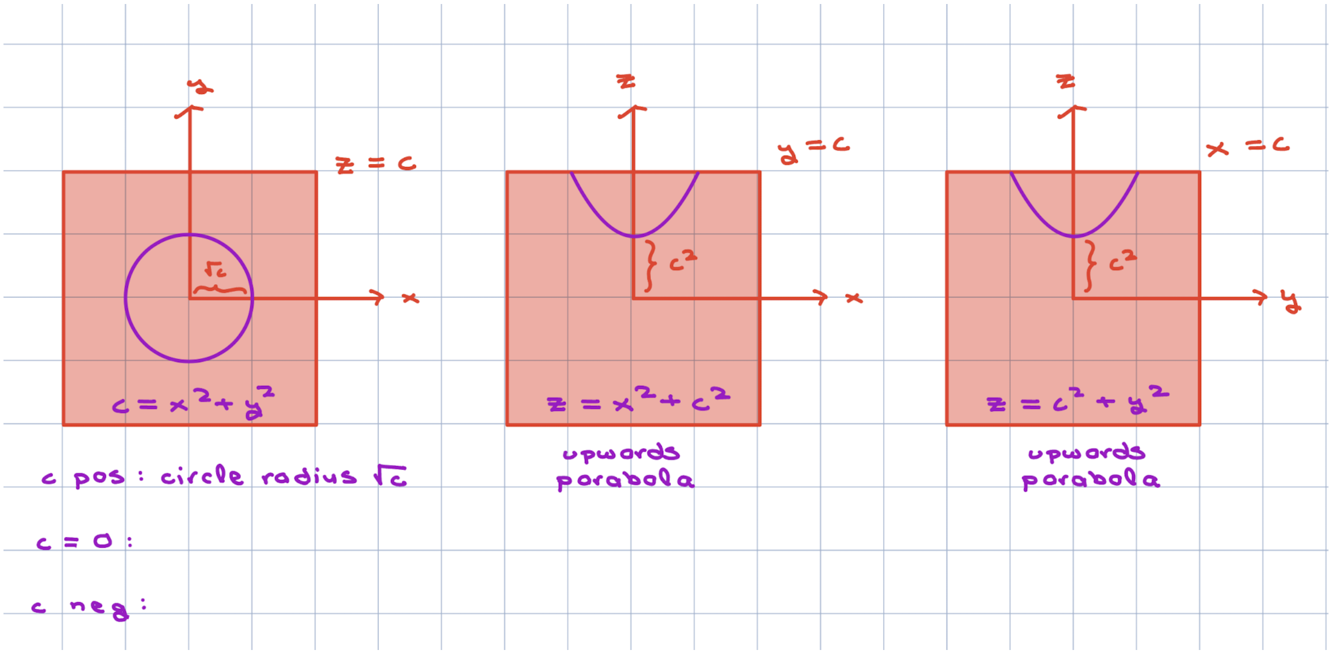




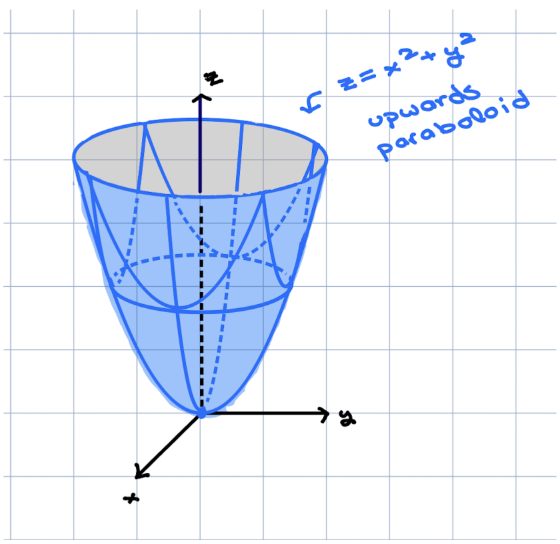
In this case, the equation of the $y = c$ slice is $x^2 + z^2 = 1$, independent of the constant c. Therefore this surface has y-**translational symmetry**, meaning you can take any point on the surface, then translate it in the y-direction (in the picture this would be left/right), and you will still be on the surface!

A **[your-favorite-curve] cylinder** is a surface obtained from taking **[your-favorite-curve]** and translating it in one direction. Hence this cylinder is a **circular** cylinder. But you could have a **parabolic**cylinder (translate a parablola) or an **elliptic** cylinder (translate an ellipse) and so on. See discussion for details.

B. **Paraboloids.** We next look at the equation $z = x^2 + y^2$ and consider its $z = c$, $y = c$, and $x = c$ slices. In Desmos.

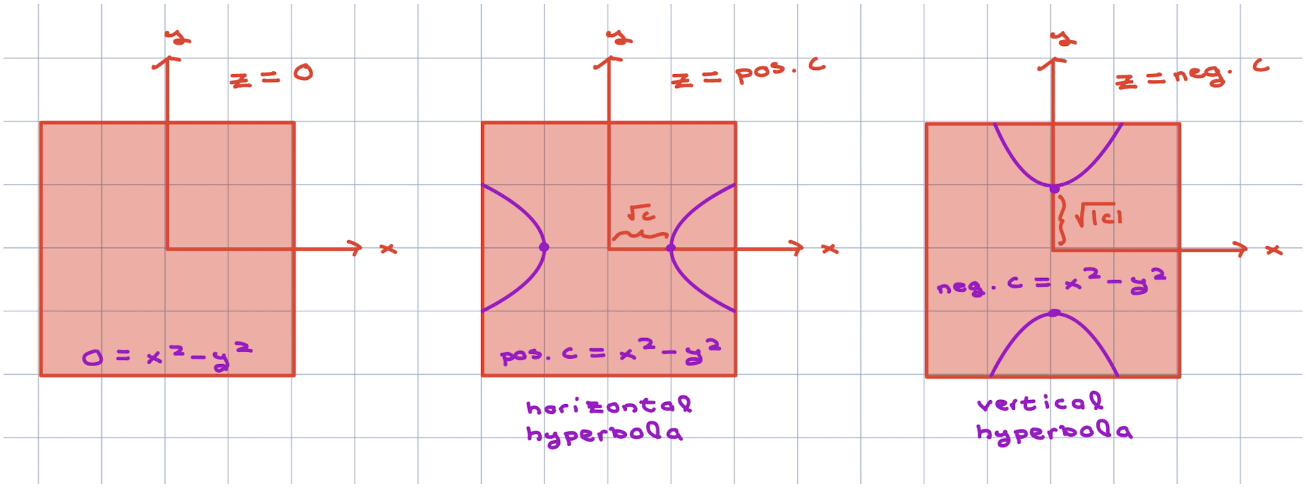


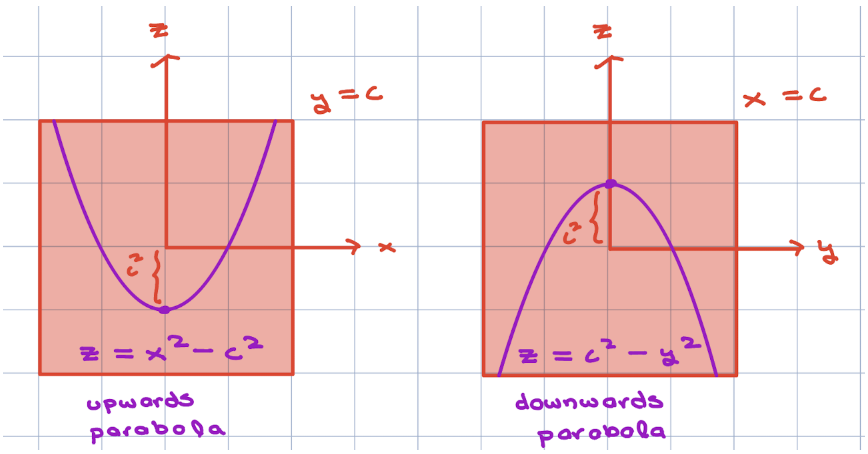
We put these together to obtain the surface it defines, which we call an upwards **paraboloid**.



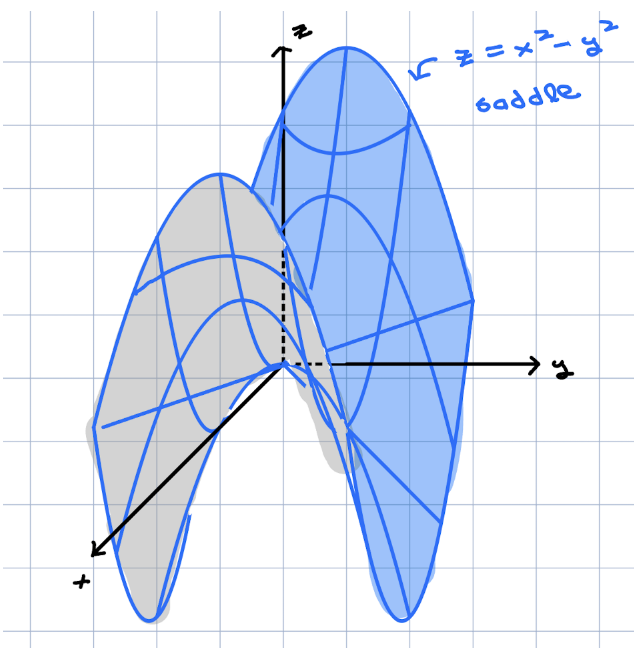
Making this sketch was time consuming. Very. Time-consuming.

C. **Saddles.** And now the equation $z = x^2 - y^2$ and its slices. In Desmos.





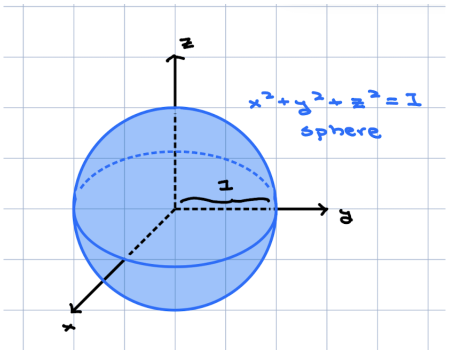
We put these together to obtain the surface it defines, which is technically called a **hyperboloic paraboloid** but is informally referred to as a **saddle**.



Can you imagine a little dude mounting this saddle on his little horse? Drawing this, again achieving some level of accuracy, was extraordinarily time-consuming. I wish I was an artist.

D. **Spheres and Hyperboloids.** The **unit sphere** is defined by the equation:

$$x^2 + y^2 + z^2 = 1$$

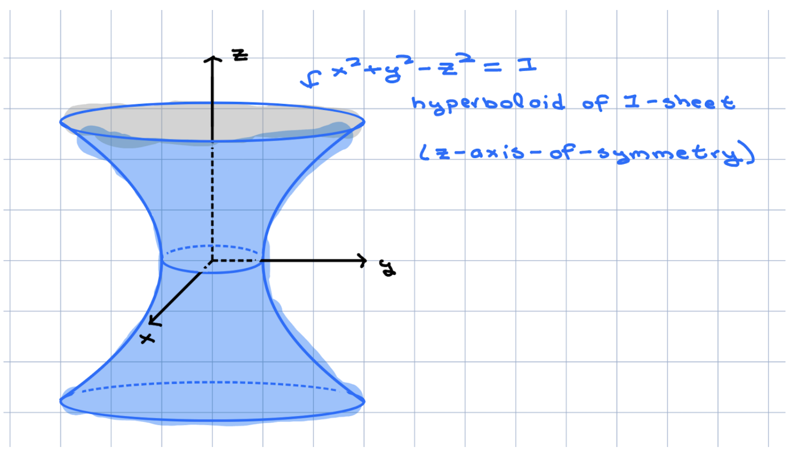


The equation $x^2 + y^2 + z^2 = 1$ is saying that the distance square of (x, y, z) from the origin is 1, which is why we obtain the unit sphere.

What if we change some of the signs in the lefthand side of the equation that defines the unit sphere? If 1 sign on the lefthand side of the sphere equation is flipped negative, as with:

$$x^2 + y^2 - z^2 = 1$$

we obtain the **hyperboloid of 1-sheet**. In Desmos.

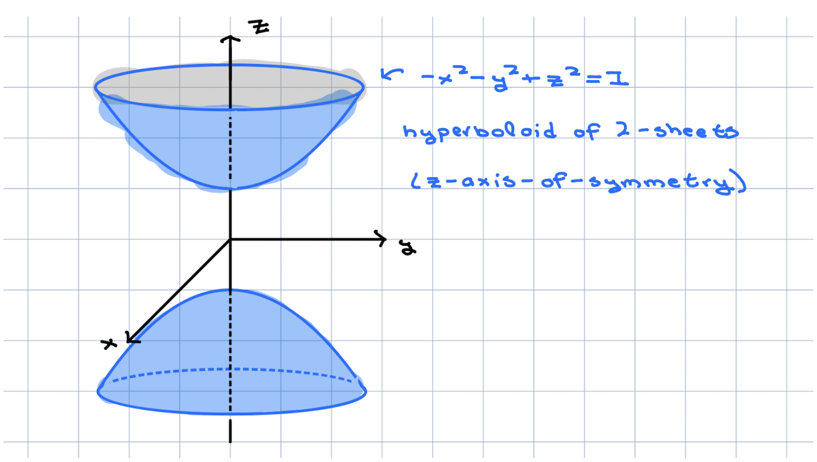


Its vertical slices are (usually) hyperbolas, and its horizontal slices are circles.

If 2 signs on the lefthand side of the sphere equation are flipped negative, as with:

$$-x^2 - y^2 + z^2 = 1$$

we obtain the **hyperboloid of 2-sheets**. In Desmos.



Its vertical slices are hyperbolas, and its horizontal slices are (usually) circles. But why does it not intersect the xy-plane? What happens if you set $z = 0$ in the equation? Compare this to setting $z = 0$ in the equation above of the hyperboloid of 1-sheet.