A. **Graphical Operations.** Surfaces can be stretched, shrunk, translated, and reflected. For example let us talk about the **ellipsoid** defined by:

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-2}{3}\right)^2 + \left(\frac{z-3}{4}\right)^2 = 1$$

in relation to the unit sphere $x^2 + y^2 + z^2 = 1$. Let's do it using Desmos.

Desmos3D is a great resource if you would like to view a surface from multiple perspectives.

Below are some surface transformations, with examples of variable replacements in the equations that define those surfaces.

transformation	example	example: change to surface
scaling	$x \mapsto \frac{x}{2}$	stretch surface by factor of 2 in x
translation	$z \mapsto z-3$	shift surface 3 units up (in z-direction)
reflection	$x \leftrightarrow z$	reflect surface across plane $x = z$

These transformations more or less preserve the shape of a surface. So, if for example, we applied them to a paraboloid, we would still call the result a paraboloid.

Here is a list of standard equations of quadric surfaces. Scaling, translating, or reflecting these surfaces does not change the below title of the surface.

Surface:	**Standard Equation**
paraboloid:	$z = x^2 + y^2$ 
saddle:	$z = x^2 - y^2$ 
ellipsoid—sphere if uniformly scaled:	$x^2 + y^2 + z^2 = 1$ 
1–sheeted hyperboloid:	$x^2 + y^2 - z^2 = 1$ 
2–sheeted hyperboloid	$-x^2 - y^2 + z^2 = 1$ 
double–cone:	$z^2 = x^2 + y^2$ 

Remember, a quadric surface is defined by an equation involving a two-variable polynomial of degree 2.

You will see the double–cone in discussion section.

Example 1. Classify the surface.

$4x^2 - 16y^2 + z^2 = -16$

This is a warning not to be too attached to merely counting the signs on one side of the equation without sparing a moment to think about the other side!

$(x+3)^2 - y + z^2 = 1$

B. **Curves.** The 1D objects in space are called **curves**. Just like for straight lines, we will describe curves with **parametrizations**.

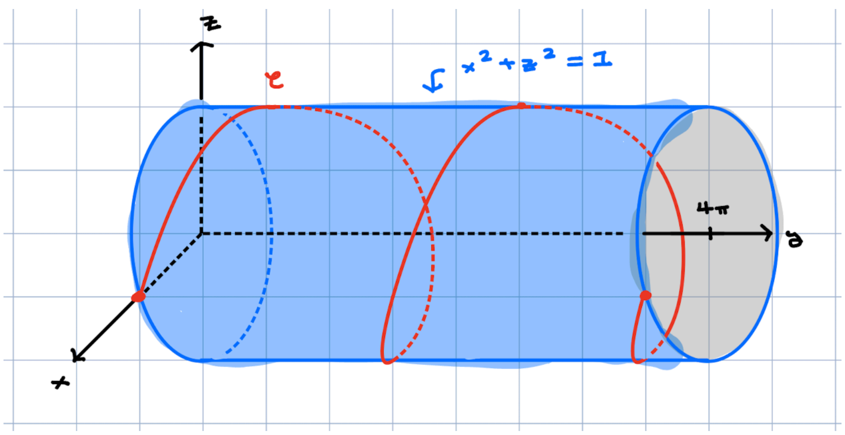
As an example, let's consider the curve $\mathcal{C}$ parametrized by:

Remember: a parametrization is a function whose outputs are the position vectors of points along the curve.

$$\vec{r}(t) = \langle \cos t, t, \sin t \rangle \quad \text{with} \quad 0 \leqslant t \leqslant 4\pi$$

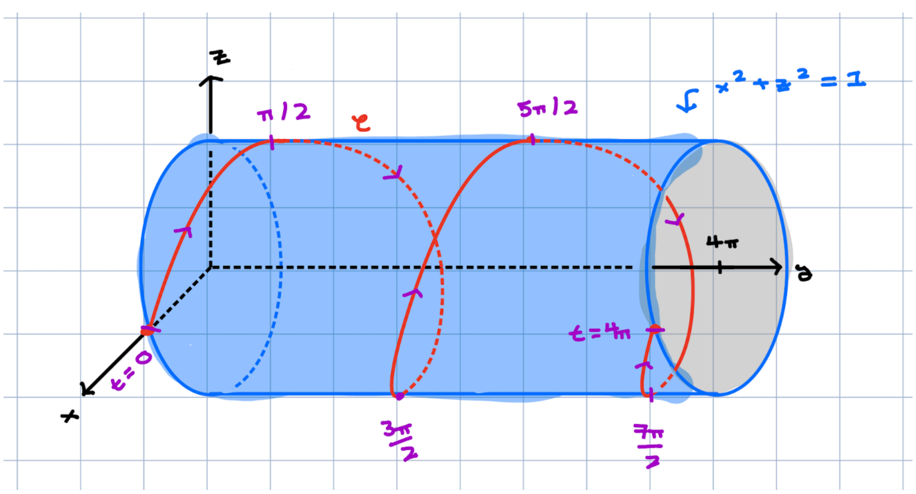
First let us verify that every point on this curve is on the cylinder $x^2 + z^2 = 1$.

Now we provide a sketch. In Desmos.



Note that the xz-coordinates circle around the z-axis, because they are $\cos t$ and $\sin t$ which literally trace out a circle!

The parametrization adds more detail then the physical curve itself.



The parametrization encodes an **orientation** (a choice of direction along the curve) as well as a **speed** (if you think of t as time. Both of these are not intrinsic to the physical curve in space, they are extra!

The parametrization $\vec{r}(t)$ is also called a **path** to distinguish it from fixed-in-space curve $\mathcal{C}$.

Example 2. Consider the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ with respective parametrizations:

$\vec{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$

$\vec{r}_2(t) = \langle 3-t, t-2, t^2 \rangle$

Do $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect? And if so, at what point?

In order to check whether the curves intersect we need to check that any point at any parameter on the first curve does not match with any point at any parameter on the second curve. To set this up as an equation, we need to make sure we consider the possibility that the parameter on the first curve is different from the parameter on the second curve.

Example 3. Parametrize the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = xy$. In Desmos.