A. **Path Derivatives.** Paths—parametrizations for curves—have speed and direction. Another word for this combination is **velocity**. This should ring a calculus bell: velocity is the derivative of position.

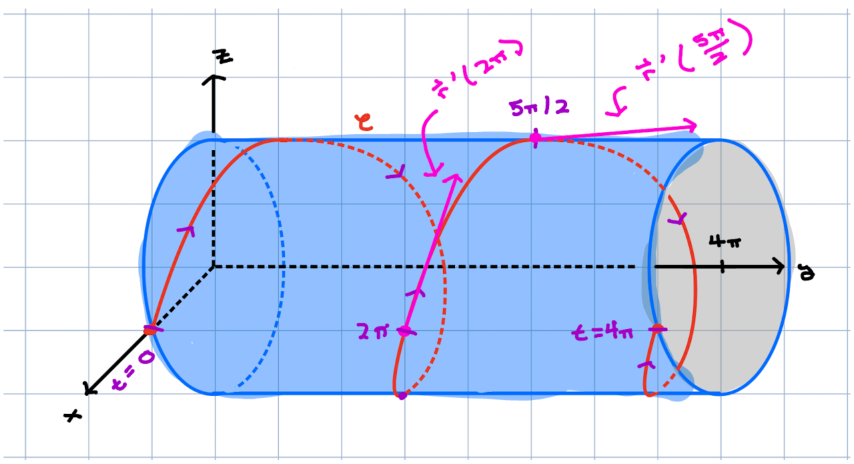
The **path derivative** (velocity vector) is found by taking the derivative of each component of a path. For example if:

$$\vec{r}(t) = \langle \cos t, t, \sin t \rangle$$

then:

$$\vec{r}\,'(t) =$$

Let us look at investigate visually. In Desmos.



When t is thought of as time and $\vec{r}(t)$ as position of a particle in space, then the path derivative $\vec{r}\,'(t)$ is also called the **velocity vector**. It points in the direction of the particle's path at time t and it length $\|\vec{r}\,'(t)\|$ is its **speed**.

Because $\vec{r}\,'(t)$ points in the direction of the particle's path along the curve, we also say that it is **tangent** to the curve.

Example 1. Let $\mathcal{C}$ be the curve parametrized by:

$$\vec{r}(t) = \langle \sqrt{t}, 2 - t, t^2 \rangle$$

Find a parametrization for the **tangent line** to the curve at the point:

$$P = (2, -2, 16)$$

Remember: the velocity vector $\vec{r}\,'(t)$ provides a tangent vector to the curve at the point on the curve at parameter t. So we will use that velocity vector as the direction vector for our line.

Example 2. In an earlier example you found that the paths:

$$\vec{r}_1(t_1) = \langle t_1, 1 - t_1, 3 + t_1^2 \rangle$$

$$\vec{r}_2(t_2) = \langle 3 - t_2, t_2 - 2, t_2^2 \rangle$$

intersect when:

$$t_1 = 1$$

$$t_2 = 2$$

Find the **angle** between the paths at this point of intersection.

By the angle between the paths, we mean the angle between their velocity vectors.

B. **Path Derivative Rules.** Derivatives come equipped with rules.

Sum:	$\left(\vec{r}_1 + \vec{r}_2\right)' = \vec{r}_1{}' + \vec{r}_2{}'$
Constant Multiple:	$\left(c\,\vec{r}\right)' = c\,\vec{r}\,'$
Dot Product:	$\left(\vec{r}_1 \cdot \vec{r}_2\right)' = \vec{r}_1{}' \cdot \vec{r}_2 \ + \ \vec{r}_1 \cdot \vec{r}_2{}'$
Cross Product:	$\left(\vec{r}_1 \times \vec{r}_2\right)' = \vec{r}_1{}' \times \vec{r}_2 \ + \ \vec{r}_1 \times \vec{r}_2{}'$

$\vec{r}_1(t)$ and $\vec{r}_2(t)$ denote differentiable paths, and c denotes a constant scalar. We write $\vec{r}_1$ and $\vec{r}_2$ without the (t) for ease of readibility.

Remember that for cross products, order matters! Changing order flips sign. Therefore the order in the cross product rule is important to pay attention to!

Example 3. Let $\vec{r}(t)$ be a differentiable parametrization for a curve that lies entirely on the sphere:

$$x^2 + y^2 + z^2 = 1$$

Show that $\vec{r}\,'(t)$ is always orthogonal to $\vec{r}(t)$.

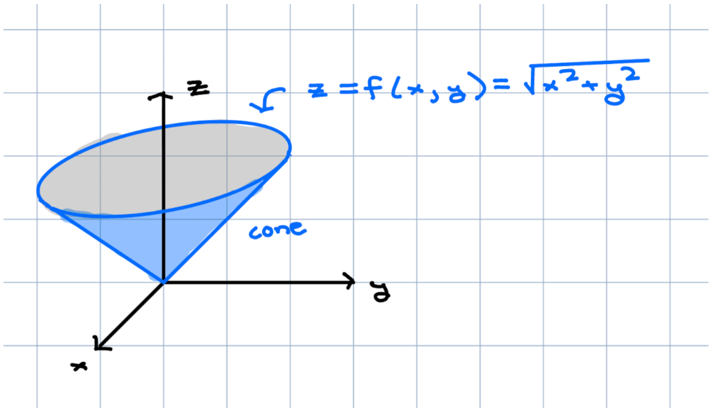
These vectors will be orthogonal when $\vec{r}\,'(t) \cdot \vec{r}(t) = 0$. Does the lefthand side not look reminiscent of a piece of the dot product rule?

C. **Multivariable Functions and Level Curves.** The functions we encounter in this course will generally involve multiple input variables. For example:

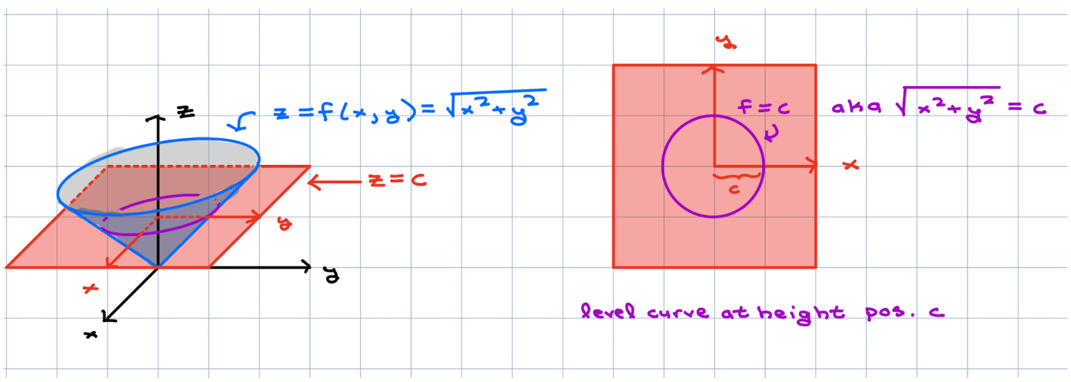
$$f(x, y) = \sqrt{x^2 + y^2}$$

To graph a function, we set the outputs equal to another variable. In this case, the graph is defined by:

$$z = \sqrt{x^2 + y^2}$$

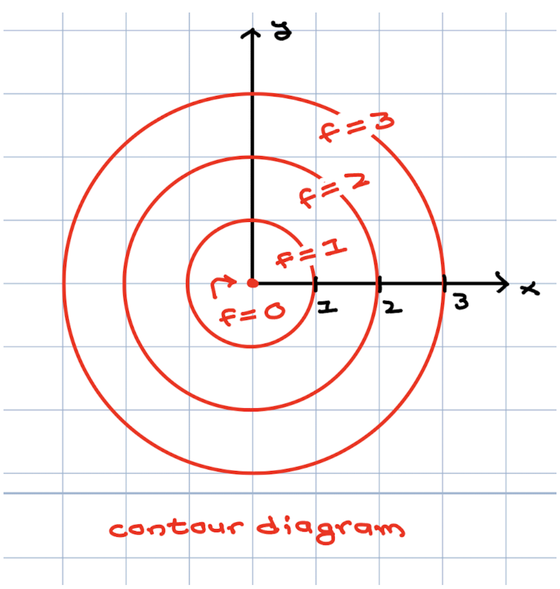


A **level set** at height c is the set of inputs with output c:



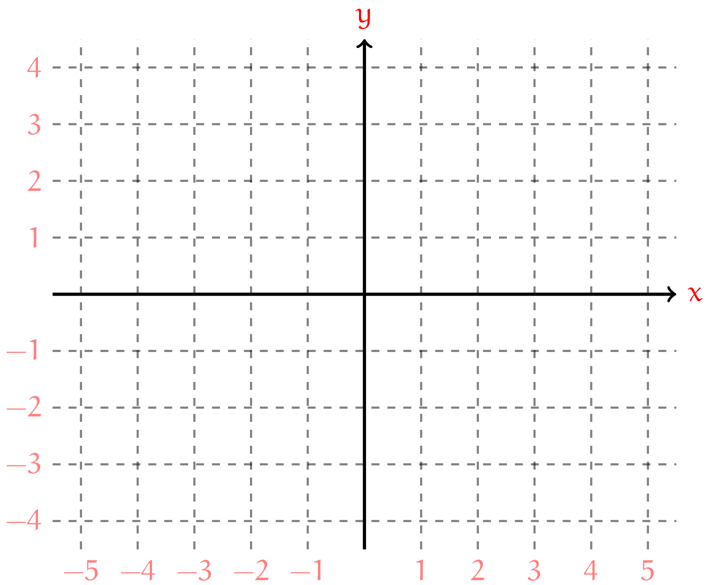
In this case, the level set is a curve, and therefore we would call it a **level curve**. Another name for a level curve is a **contour**.

If we arrange many level curves together we get a **contour diagram**:



Example 4. Sketch a contour diagram for:

$f(x, y) = y - 2x$



The graph of this function is defined by the equation $z = y - 2x$. This is the equation of a plane!