A. **Higher–Order Derivatives.** There is nothing stopping us from taking repeated partial derivatives.

And if there is nothing stopping us, then why should we stop. Just keep at it until you draw your concluding breath on this small planet.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} =$$

$$f_x = \frac{\partial f}{\partial x} =$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} =$$

$f = xy^2 + e^y$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} =$$

$$f_y = \frac{\partial f}{\partial y} =$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} =$$

Clairaut's Theorem: Where a function has continuous partial derivatives of all orders, the order in which we execute partial differentiation does not matter.

The **order** of a hihger-order partial derivative is the number of $\partial/\partial\star$'s it took to get you there. So $\partial^2 f/(\partial x \partial y)$ is a 2nd-order partial derivative.

Virtually every function we encounter in this course will have continuous partial derivatives of all orders, at least where its partial derivatives of all orders are defined. So you can apply Clairaut's Theorem without further comment.

We have not even really talked about continuity of multivariable functions, so you are certainly not expected to verify it. You may just assume that we are working with nice-enough functions that, when you need Clairaut's Theorem, you may apply it. :)

So for example:

$f_{yyx} =$

Example 1. Find $f_{xyzzyxz}(x, y, z)$ where:

$$f(x, y, z) = \frac{z^2 e^{xy^2+\sin(x+y)}}{x}$$

Only try a direct approach if you like the idea of self-inflicted pain.

B. **Chain Rule.** Consider the function:

$$z = x^2 y$$

but suppose x and y further depend on variables s and t.

$$\begin{cases} x = st \\ y = e^{st} \end{cases}$$

This gives us the following tree of dependencies:

This looks just like the tree in my backyard. I need to have it dealt with.

Goal: find $\partial z/\partial s$ and $\partial z/\partial t$ **without** first writing z in terms of s and t.

Why tie our hands in this way? Restrictions force us to be creative and think in new ways. Try imposing some restrictions in your personal life.

> **Chain Rule.** To find $\partial z/\partial\star$ you should identify each path in the tree starting at z and ending at $\star$, convert that path into a product of partial derivatives, and then add the results.

The description to the left is perhaps painfully succinct, such that it has lost all meaning. First: we can convert each edge in the tree to a partial derivative. Like the edge $z \longrightarrow x$ can be converted to $\partial z/\partial x$.

So, in our example:

$$\frac{\partial z}{\partial s} =$$

Then: the paths from z to s in our tree are:

$z \longrightarrow x \longrightarrow s$ converts to $(\partial z/\partial x) \cdot (\partial x/\partial s)$

$z \longrightarrow y \longrightarrow s$ converts to $(\partial z/\partial y) \cdot (\partial y/\partial s)$

$\partial z/\partial s$ is the sum of the terms on the right.

$$\frac{\partial z}{\partial t} =$$

Why does the chain rule work? Well it helps to think of partial derivatives as rates of change. If s increases by 1 unit, then x is increased $(\partial x/\partial s)$ times (1) units. And if x is increased by $\partial x/\partial s$ units, then z is increased by $(\partial z/\partial x)$ times $(\partial x/\partial s)$ units. This is where the $z \longrightarrow x \longrightarrow s$ part of the chain rule comes from.

Example 2. Let $z = f(x, y)$ be a differentiable function and let:

$$\begin{cases} x = s + t \\ y = s - t \end{cases}$$

Show that:

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Note here that we are not told what $f(x, y)$ is, beyond the fact that it is differentiable. We need to verify that the given equality is true, independent of what $f(x, y)$ is.

C. **Chain Rule and Gradient.** We will view the chain rule a different way, assuming the inside function is a **path**. Consider:

$$f(x, y, z) = xe^{y/z}$$

but the variables x, y, and z further depend on variable t.

$$\begin{cases} x = t^2 \\ y = -t \\ z = 2t \end{cases}$$

which we instead think about as **path**:

$$\vec{r}(t) =$$

We now write the following using dot products:

$$= \frac{df}{dt} =$$

The **gradient** of $f(x, y, z)$ is:

$$\vec{\nabla} f(x, y, z) =$$

The chain rule says that:

$$\frac{d}{dt}\left[f\left(\vec{r}(t)\right)\right] =$$

Example 3. Let $f(x, y) = \cos x \ln y$ and let $\vec{r}(t)$ be a differentiable path so:

$$\vec{r}(0) = \langle 0, 1 \rangle$$

$$\vec{r}\,'(0) = \langle -1, 1 \rangle$$

Let $g(t) = f(\vec{r}(t))$ and find $g'(0)$.