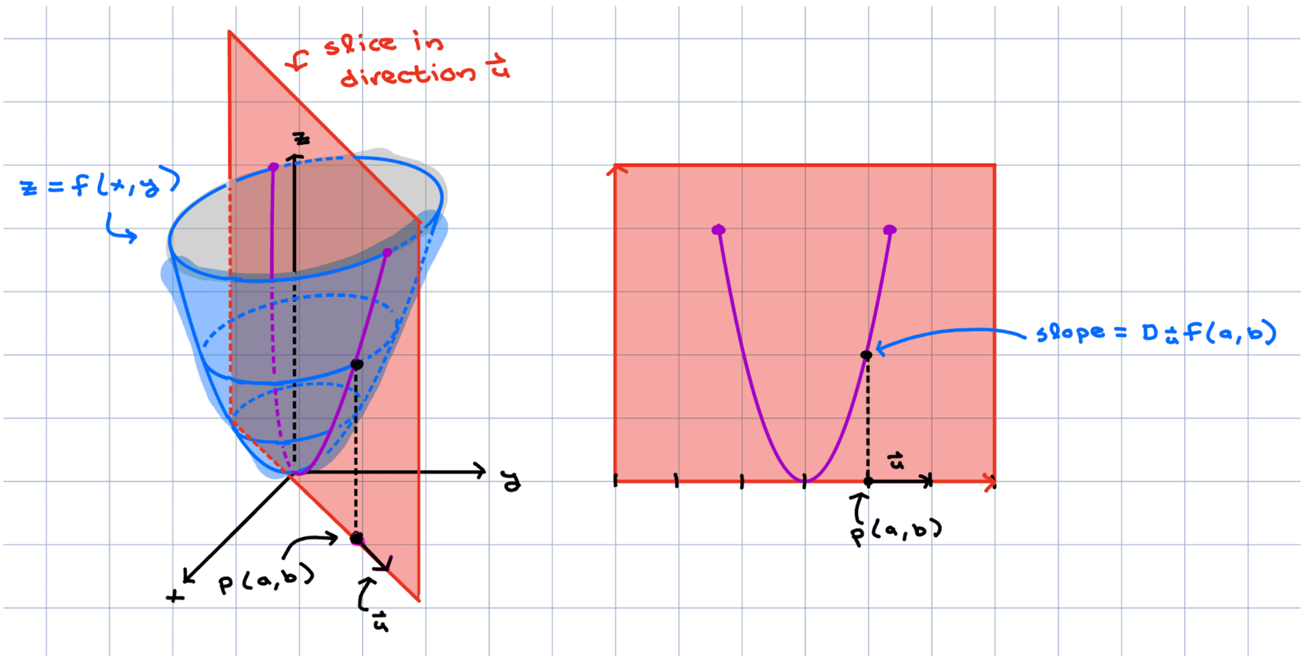
A. **Directional Derivatives.** We can measure rates of change in directions besides the coordinate directions. Consider function $f(x, y)$ and a point $P(a, b)$ along with a unit direction $\vec{u}$.

The rates of change in the coordinate directions are the partial derivatives.

By a unit direction $\vec{u}$ we mean a unit vector $\vec{u}$.



The rate of change of f, beginning at input P and moving in unit direction $\vec{u}$, is called the $\vec{u}$**–directional derivative** of f at P and is denoted $D_{\vec{u}}f(P)$.

To find a formula for the directional derivative, we parametrize the $\vec{u}$–axis:

$\vec{r}(t) =$

and then calculate the rate of change of f along this parametrization, at P:

$$\frac{d}{dt}\left[f(\vec{r}(t))\right]\Big|_{t=0} =$$

A formula for the directional derivative is $D_{\vec{u}}f(P) =$

Here $\vec{u}$ is required to be a unit vector.

Example 1. Calculate the directional derivative of $f(x, y) = xye^y$ at $P(-2, -1)$ in the direction determined by $\vec{v} = \langle -1, -1 \rangle$.

Because $\vec{v}$ is not a unit vector, we first need to convert it to one. Directional derivatives are only defined for unit vectors. If you use the formula for a non–unit vector, then what you are computing is not a slope.

B. **Optimizing the Directional Derivative.** Directional derivatives tell us the rate of change of a function in different unit directions. What if we wanted to find the direction where that increase was most positive or negative?

We consider the angle θ between a unit direction $\vec{u}$ and the gradient $\vec{\nabla} f(P)$.

$D_{\vec{u}} f(P) =$

The directional derivative $D_{\vec{u}} f(P)$ is:

- **maximized** in direction $\vec{u} =$ with value $D_{\vec{u}} f(P) =$
- **minimized** in direction $\vec{u} =$ with value $D_{\vec{u}} f(P) =$
- **zero** in directions $\vec{u}$

Example 2. T measures temperature (in degrees fahrenheit) and (x, y) measures coordinates (in cm) on a hot pan.

An ant is at point $P(2, 1)$ on the pan. If the ant:

- moves right, then the temperature increases at a rate of 12 degrees/cm
- moves up, then the temperature decreases at a rate of 5 degrees/cm

Rates of change in particular directions are measured by directional derivatives. When that direction is to the right, this is an x-partial derivative. When that direction is up, this is a y-partial derivative.

(a) Find $\vec{\nabla}T(2, 1) =$

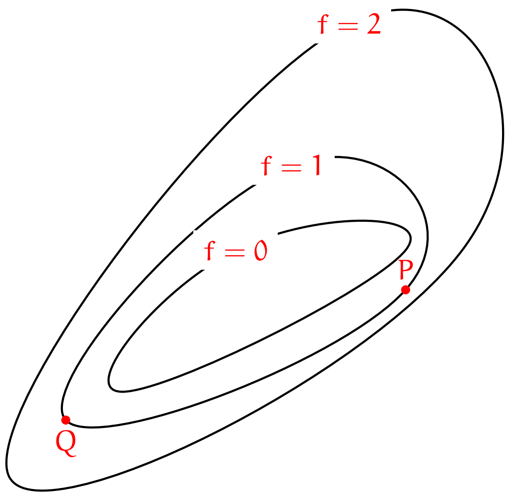
(b) At what rate does the temperature change if the ant moves from $P(2, 1)$ in the unit direction $\vec{u} = \langle 3/5, 4/5 \rangle$.

(c) In which unit direction $\vec{u}$ from $P(2, 1)$ should the ant move to cool off most rapidly? What is the rate of change of temperature in that direction?

(d) In which units direction $\vec{u}$ from $P(2, 1)$ could the ant move to remain roughly at the same temperature.

C. **Gradients and Level Sets.** Consider the contour diagram for the function $f(x, y)$ depicted below. We will estimate of $\vec{\nabla} f(Q)$ and $\vec{\nabla} f(P)$.

Remember that each contour is called a **level curve** or **level set**.



Remember: the gradient points orthogonal to the direction of "sameness" of f. Further it points in the direction of maximum increase of f, and its length is determined by how quickly f changes.

The gradient $\vec{\nabla} f(P)$ is, when rooted at P, orthogonal to the level set of f that contains P, and it points towards "nearby" level sets with higher value.