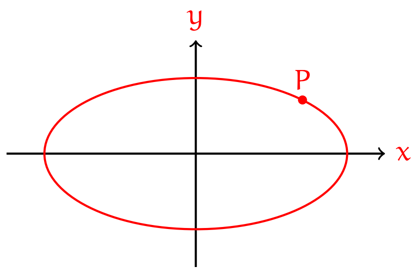
Example 1. Consider the ellipse: $x^2 + 4y^2 = 9$.



(a) Interpret this as a level curve.

(b) Parametrize the **normal line** to the ellipse at $P = (2, 1)$.

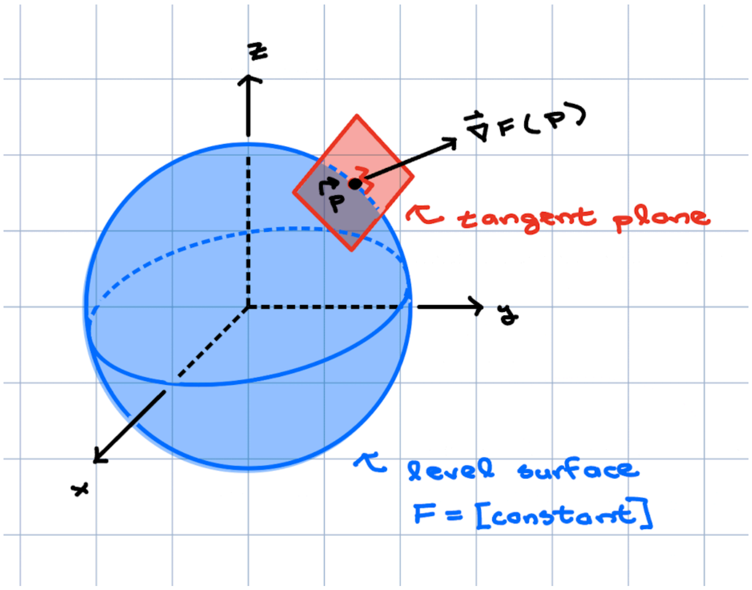
The normal line is the line through the point that is orthogonal (aka normal) to the curve.

(c) Use the gradient to find an equation for the **tangent** line to the ellipse at P.

In 3D, if you have a normal vector $\vec{n}$ and a point P, you can find the equation of a plane using $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$. If the vector and point are in 2D, then what you will obtain is the equation of a line instead of a plane.

A. **Gradients and Tangent Planes.** Consider the level surface $F(x, y, z) = C$ and the tangent plane at point P on that level surface.

We know that gradients are orthogonal to level sets, and therefore $\vec{\nabla} F(P)$ is orthogonal to this level surface, and consequently is a normal vector for the tangent plane to this surface.



The tangent plane to the level surface $F(x, y, z) = C$ at point P is defined by:

You might be wondering: have we not already provided a formula for tangent planes? Well, yes. But only in the special case of a graph $z = f(x, y)$, in which case the tangent plane at $x = a$ and $y = b$ has equation:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

In fact we can relate this to the equation on the left. Rewrite $z = f(x, y)$ as:

$$f(x, y) - z = 0$$

to realize that this surface is a level set of $F(x, y, z) = f(x, y) - z$. Now apply the formula on the left at $P(a, b, f(a, b))$ to obtain the same plane equation given above.

The formula on the left applies in more cases, like when the surface is not a graph of a function of x and y.

Example 2. Consider the surface defined by $xyz = 8$.

(a) Find an equation for the tangent plane to this surface at $P(2, 2, 2)$.

(b) Find all points on this surface, $xyz = 8$, where the tangent plane is parallel to the plane below.

$\mathcal{P} : 4x + 2y + z = 100$

Two planes are parallel if their normal vectors are parallel. Their normal vectors are parallel if one is a scalar multiple of the other. One way to investigate this is to set up the equation:

$(\text{plane 1 normal}) = \lambda(\text{plane 2 normal})$

where λ (read "lambda") represents an undetermined scalar.