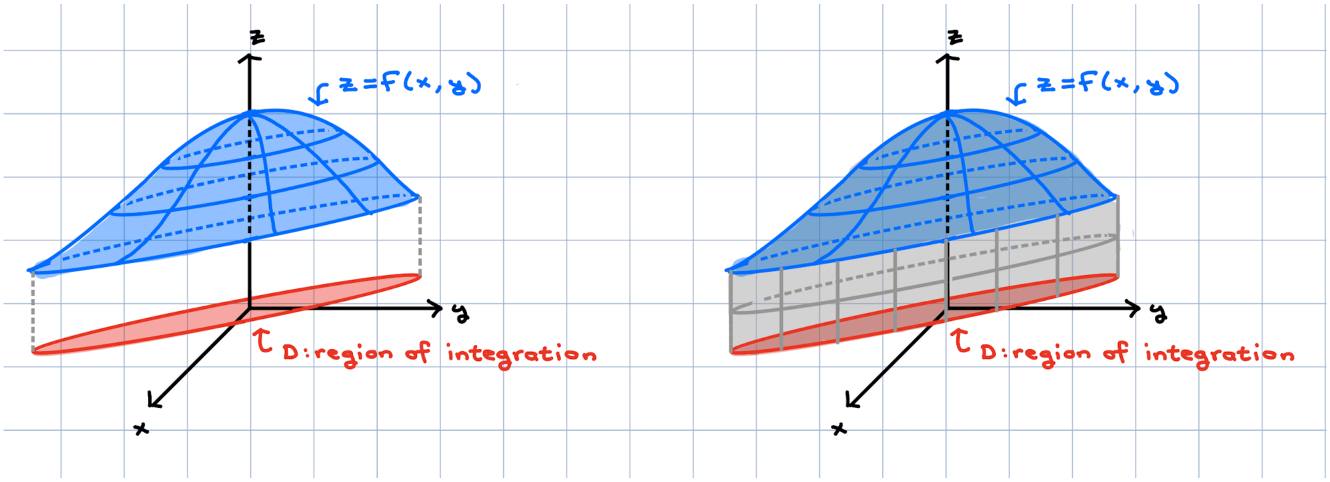
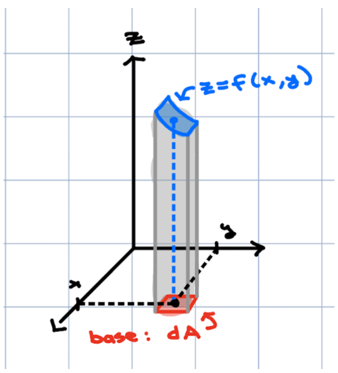
A. **Double Integrals.** Let's investigate the **vertical** volume between a graph $z = f(x, y)$ and a **region of integration** D in the xy-plane.



We decompose this volume up into **infinitesimally** thin rectangular boxes.



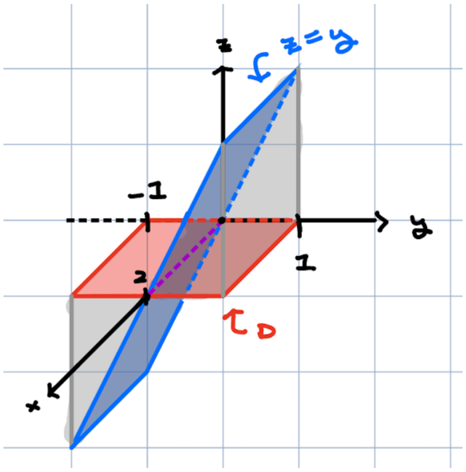
An integral is a way of taking an infinite sum. Since we have infinitesimally thin rectangular boxes, we will need to take an infinite sum of their volumes, one for each infinitesimal bit of area dA in the region of integration. Because these bits of area are two-dimensional, we use two integral signs instead of one.

The **double integral** of $f(x, y)$ over region of integration D is:

and equals the **signed** vertical **volume** between the graph $z = f(x, y)$ and the region D in the xy-plane.

We use the word **signed** here, because the volume obtained from a double-integral can be negative. This can be seen by looking at how we calculated the volume of each box. The height is $f(x, y)$, but that height can be negative, which occurs when the graph is below the xy-plane. So, signed volume here means that volume below the xy-plane will count as negative, while voluem above the xy-plane will count as positive.

Example 1. Let D be the rectangle $0 \leqslant x \leqslant 2$ and $-1 \leqslant y \leqslant 1$ and let $f(x, y) = y$.



Find: $\iint_D f(x, y)\, dA$

This rectangle is denoted more compactly as $[0, 2] \times [-1, 1]$.

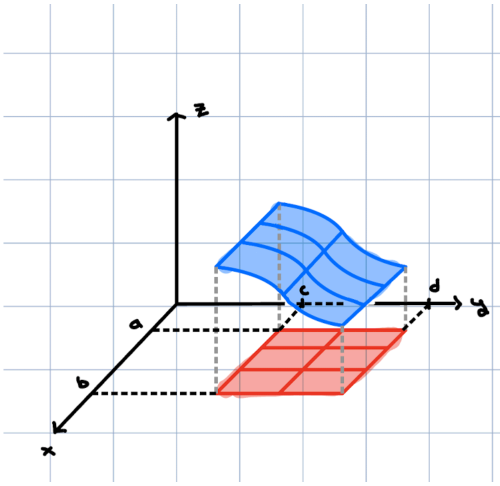
We say f is **odd in** x if $f(-x, y) = -f(x, y)$ and is **odd in** y if $f(x, -y) = -f(x, y)$.

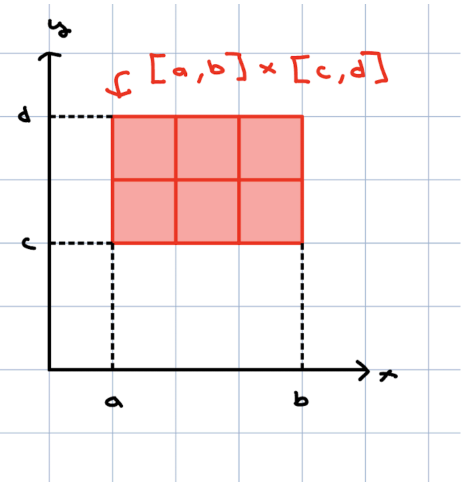
We say that a region is **symmetric in** x if whenever (x, y) is in the region, then $(-x, y)$ is also in the region. We say that a region is **symmetric in** y if whenever (x, y) is in the region, then $(x, -y)$ is also in the region.

An integral of a function that is odd in a variable over a region that is symmetric in that variable will equal 0 due to signed volumes cancelling out.

B. **Double Integrals Over Rectangular Regions.** Of special interest are double integrals where the region of integration is a rectangle.

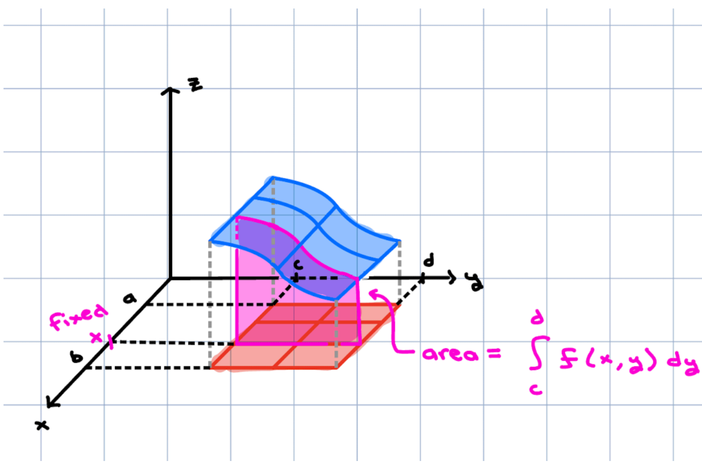
The reason these regions are of special interest, is because they are the simplest regions to deal with.

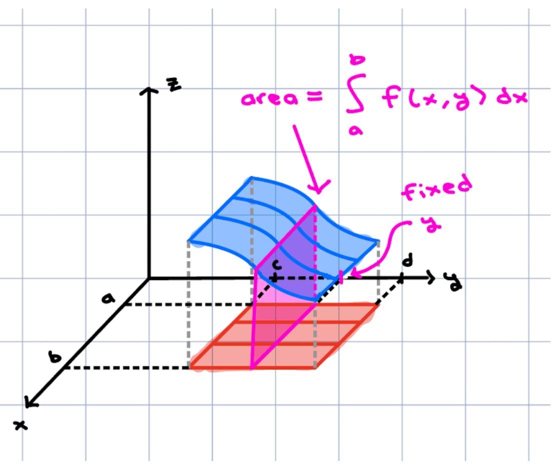




We can compute the double integral using **iterated** integrals.

Iterating integrals means to perform multiple integrals, one after the other.





An integral of $f(x, y)$ over the rectangular region $[a, b] \times [c, d]$ can be computed using iterated integrals as:

$$\iint_D f(x,y)\ dA =$$

The inner integral calculates the area of a slice, and the outer integral adds up all the areas of the slices, in total yielding the volume.

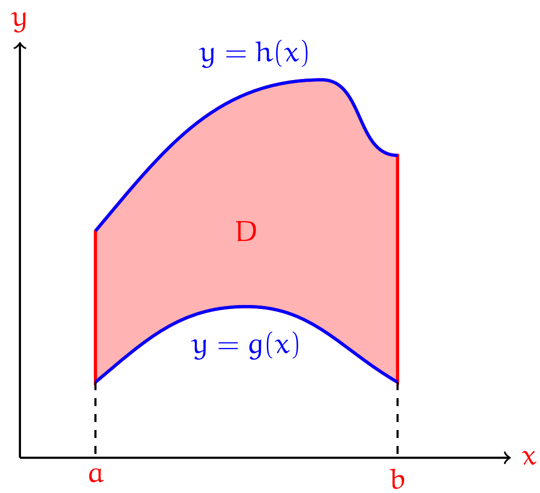
Example 2. Find the volume of the region bounded above by:

$$z = xe^{xy}$$

and below by the rectangle $D = [1,2] \times [2,3]$ in the xy-plane, using an iterated integral in order $dy\,dx$.

C. **Double Integrals Over Other Regions.** Let's show how to find double integrals over other regions of integration D.

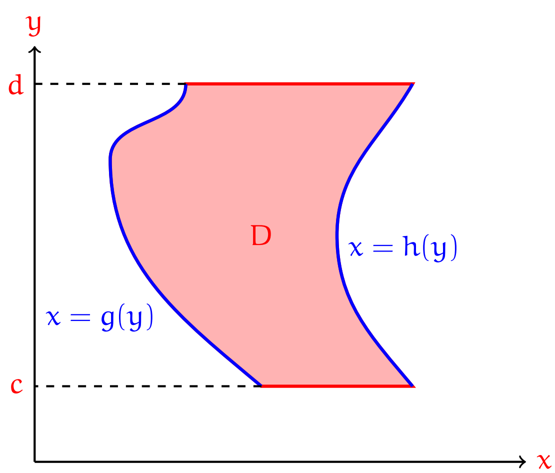
Region of Integration: Vertical Space Between Graphs. Let D be the vertical space between two graphs of $y = \text{function}(x)$ over an x-interval $[a, b]$.



$$\iint_D f(x,y)\ dA =$$

More precisely the region in question consists of all (x, y) such that $a \leqslant x \leqslant b$ and $g(x) \leqslant y \leqslant h(x)$.

Region of Integration: Horizontal Space Between Graphs. Let D be the horizontal space between two graphs $x = \text{function}(y)$ over a y-interval $[c, d]$.



$$\iint_D f(x,y)\ dA =$$

More precisely the region in question consists of all (x, y) such that $c \leqslant y \leqslant d$ and $g(y) \leqslant x \leqslant h(y)$.