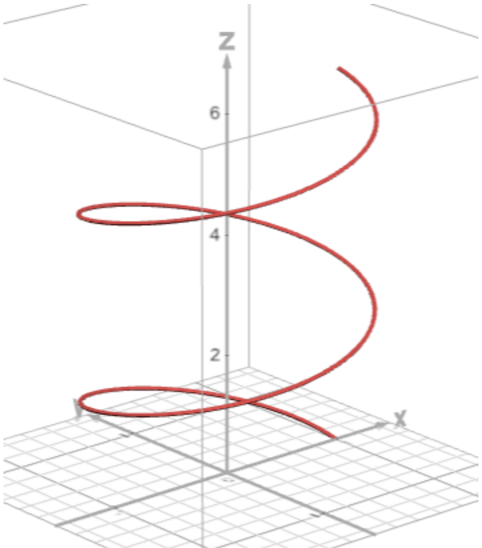
A. **Arclength.** If we add up all the infinitesimal bits of arclength, then we simply get the length of the path.

The **arclength** of a path $\vec{r}(t)$ with $a \leqslant t \leqslant b$ is:

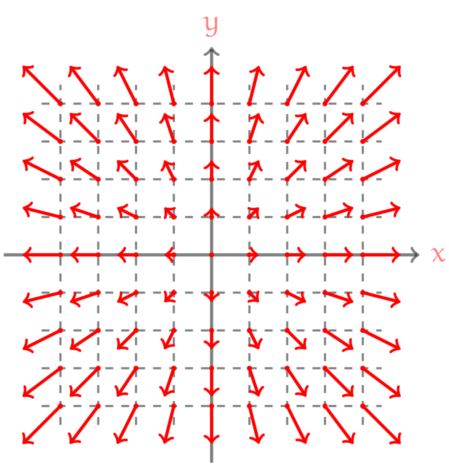
Specifically, if $\vec{r}(t)$ indicates the position of a particle at time t, then the arclength is the distance travelled by the particle.

Example 1. Find the arclength of the path:

$\vec{r}(t) = (\cos(2t), \sin(2t), t)$ with $0 \leqslant t \leqslant 2\pi$



B. **Vector Fields.** Imagine water flowing in the plane. At each point in the plane of water, you can indicate the velocity of water flow at that point. Let's visualize using Desmos.



In this picture, the flow of water is away from the origin, and the further the water is from the origin, the faster it flows. This could for example be us looking down on a sloped fountain, with water source at the origin, and the slope of the fountain becoming steeper the further from the origin we get.

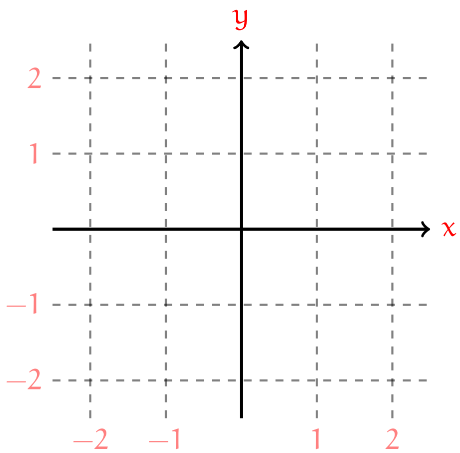
This is called a **vector field**. This one is specifically described by the **function**:

$$\vec{F}(x, y) = \left\langle P(x, y), Q(x, y) \right\rangle = \left\langle \frac{x}{4}, \frac{y}{4} \right\rangle$$

Generally a vector field is a function $\vec{F}(\vec{x})$ where the inputs and outputs are both vectors of the same dimension. The vector field sketched here is two–dimensional. A function like $\vec{F}(x, y, z) = \frac{1}{4}\langle x, y, z\rangle$ is an analagous three–dimensional vector field.

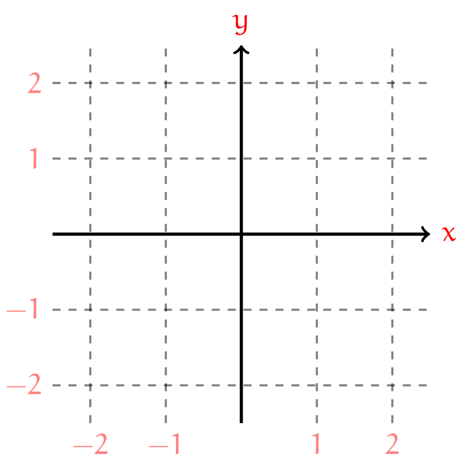
Example 2. Sketch the vector field.

(a) $\vec{F}(x, y) = \frac{1}{2}\langle 1, 1\rangle$. In Desmos.

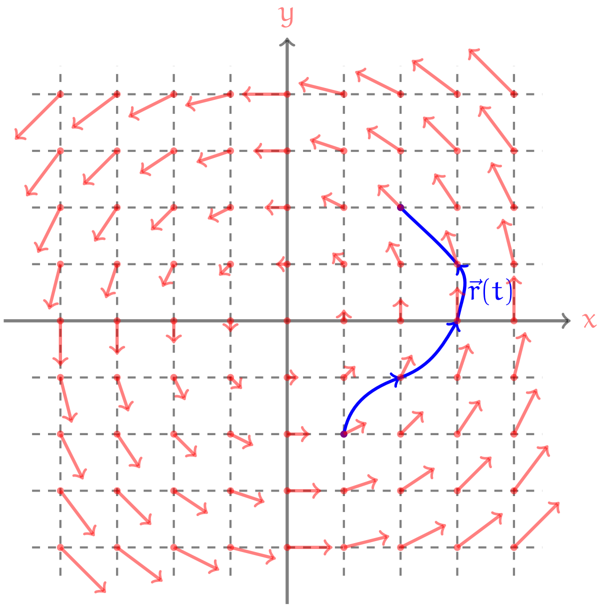


(b) $\vec{F}(x, y) = \frac{1}{5}\langle -y, x\rangle$. In Desmos.

The vector $\langle -y, x\rangle$ is the counterclockwise rotation of the vector $\langle x, y\rangle$ by 90 degrees. It is easy to see that this pair of vectors is orthogonal, by verifying their dot product is 0.

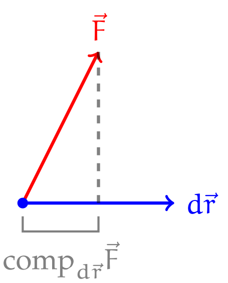


C. **Vector Line Integrals.** We will next talk about vector field line integrals. Consider this path $\vec{r}(t)$ in a fluid velocity vector field $\vec{F}$. Something similar in Desmos.



We will attempt to measure the rate of fluid flowing along the curve. At each segment $d\vec{r}$ along the curve we measure the rate of fluid flowing along that segment.

This rate will be measured in [units of area] per [unit of time]. Along each infinitesimal segment $d\vec{r}$ of the curve, the rate of fluid flowing is the product of the velocity of the fluid, in [length] per [time], by the length ds along that segment.



[rate of fluid flow **F** along **r**] $=$

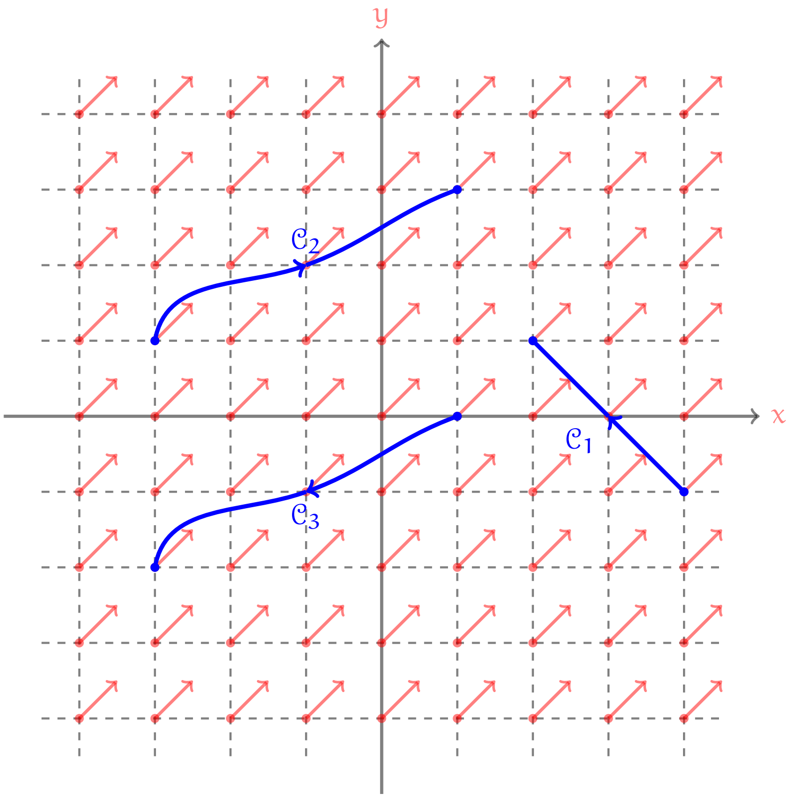
Remember that the formula for the component of vector $\vec{w}$ along vector $\vec{v}$ is $\text{comp}_{\vec{v}}(\vec{w}) = (\vec{v} \cdot \vec{w})/\|\vec{v}\|$

The the **vector line integral** of $\vec{F}$ along path $\vec{r}(t)$ with $a \leqslant t \leqslant b$ is:

$$\int_{\vec{r}} \vec{F} \cdot d\vec{r} =$$

An alternative notation for the vector line integral uses $d\vec{s}$ instead of $d\vec{r}$. It looks like $\int_{\vec{r}} \vec{F} \cdot d\vec{s}$.

Example 3. Using the vector field $\vec{F}$ sketched below, decide whether the integral is positive, negative, or zero.



Technically we need to select a parametrization in order to execute the vector line integral, but as we mention below the choice of parametrization does not matter, except for the orientation.

$$\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} =$$

$$\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} =$$

$$\int_{\mathcal{C}_3} \vec{F} \cdot d\vec{r} =$$

> An **orientation** of a curve is a choice of direction along the curve.

If the orientation of the curve involved **vector** line integral is reversed, then the value of the line integral has its sign flipped. Beyond orientation, the value of a vector line integral does not depend on the parametrization of the curve.

This is distinguished from **scalar** line integrals, where orientation does **not** matter.