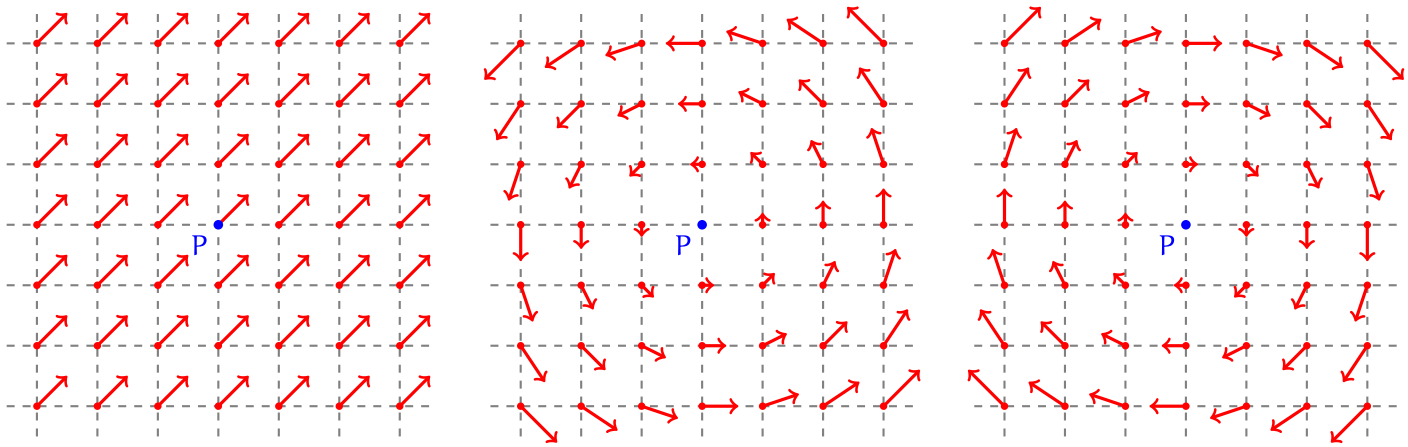
Example 1. For each vector field $\vec{F}$, decide whether $\operatorname{curl}\vec{F}(P)$ is $\mathbf{0}$, points into the page, or points out of the page.



A. **Curl and Conservative Vector Fields.** We mentioned before that conservative vector fields do not feature any circulation. Let's look once more in Desmos.

> Every conservative vector field is **irrotational**, meaning: $\operatorname{curl}\vec{F} =$

We use this idea to show that the following vector field is **not** conservative:

$\vec{F}(x, y) = \langle -y, x\rangle$

Technically the vector field $\vec{F}$ should be continuously differentiable for this to hold. This is because it relies on the Clairaut's theorem, which concludes that the order of partial differentation does not matter, under certain simple assumptions.

For a two–dimensional vector field:

$\vec{F}(x, y) = \langle P, Q\rangle$

to compute the curl we treat the third component as 0, and thus:

$\operatorname{curl}\ \vec{F}(x, y) = \operatorname{curl}\langle P, Q\rangle = (Q_x - P_y)\vec{k}$

so that a two–dimensional vector field is irrotational if and only if:

$Q_x = P_y$

Example 2. Show that the following vector field is irrotational where defined:

$$\vec{F}(x, y) = \frac{-y\vec{\imath} + x\vec{\jmath}}{x^2 + y^2}$$

Recall the simplified formula for curl of a two–dimensional vector field:

$\text{curl } \vec{F}(x, y) = \text{curl } \langle P, Q \rangle = (Q_x - P_y)\vec{k}$

The vector field to the left was the interesting vector field we had explored before. We found it is **not** conservative, because it did not have path–independent line integrals, as its line integral over the counterclockwise unit circle was not zero, it was 2π! Remember that conservative vector fields are supposed to integrate to 0 over closed curves. Anyways… this vector field is the classic example of a vector field that is irrotational, but not conservative. Scary.

In fact, it can be shown that a potential f for $\vec{F}$ on the region $x \neq 0$ is:

$$f(x, y) = \arctan\left(\frac{y}{x}\right)$$

More generally at any point, except at the origin, we can select:

$f =$

Nonetheless, as we have seen, $\vec{F}$ is **not** conservative. How do we reconcile this?

The issue is that θ cannot be defined continuously and unambigously simultaneously at all points in the xy–plane excluding the origin. Let's view the situation in Desmos.

B. **Simple–Connectedness.** The previous example showed how an irrotational vector field is not necessarily conservative. Here's what is true.

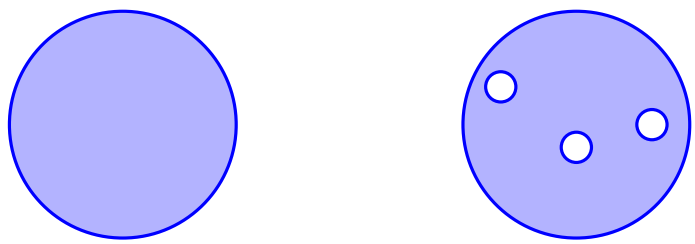
The idea behind continuous deformation is deforming the curve via bending, stretching, and shortening, but without cutting or gluing!

If $\vec{F}$ is **irrotational** on a region, then if two curves, while keeping their start point and end point fixed, can be **continuously deformed** within the region to one another, then $\vec{F}$ will integrate to:

Let's use Desmos to make sense of this.

A region is **simply–connected** if **every** pair of curves in the region with fixed start and end can, while keeping the start point and end point fixed, be continuously deformed to one another. In 2D this intuitively means:

This intuitive explanation does **not** apply to 3D. See Desmos for an example of a simply–connected region with a hole.



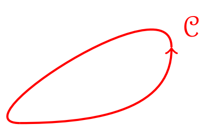
Simply–Connected And Irrotational $\implies$ Conservative.

If $\vec{F}$ is defined and irrotational on a simply–connected region, then $\vec{F}$ will have:

which turns out is enough to guarantee $\vec{F}$ is conservative on that region.

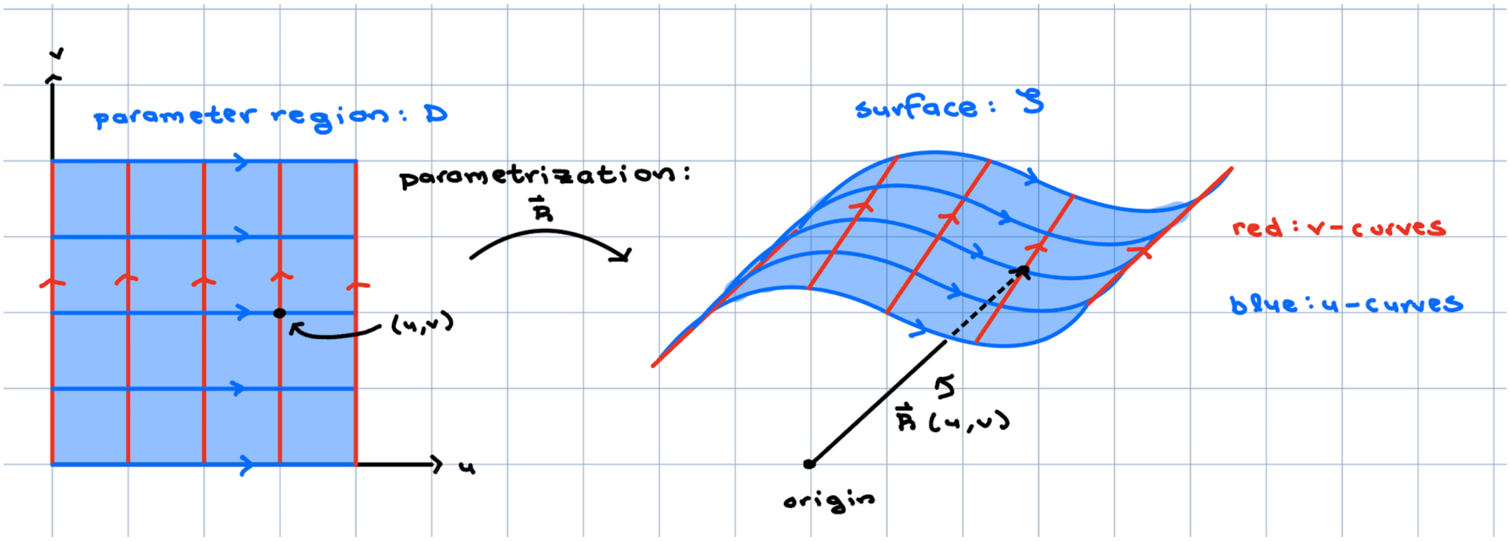
Example 3. Let $\mathcal{C}$ be the curve in the xy–plane sketched below and find:

$$\oint_{\mathcal{C}} e^{\cos(e^x)}\,dx + \sin(\sin(e^{\sin y}))\,dy$$

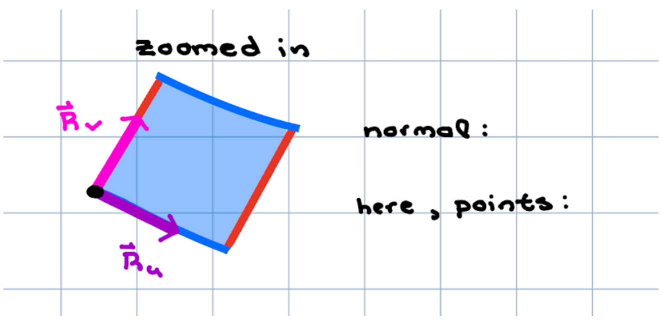


C. **Parametrized Surfaces.** We have engaged with curves for a while now. We now upgrade dimension and consider surfaces. We parametrize a surface $\mathcal{S}$ using a vector–valued function $\vec{R}(u, v)$ of two parameters u and v, where those parameters come from a specified parameter region D in the uv–plane. Let's view the situation in Desmos.

One example of a surface is a plane, with which we have in fact played around.



Our next task is to locate tangent and normal vectors to the surface.

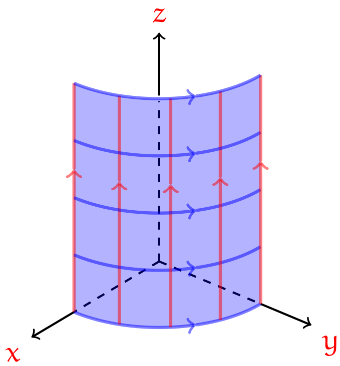


Here the partial derivatives $\vec{R}_u$ and $\vec{R}_v$ are tangent to the u–curves and v–curves respectively. To obtain a normal to the surface you would utilize a cross–product of tangent vectors.

A choice of direction for normals to the surface is called an **orientation** for the surface.

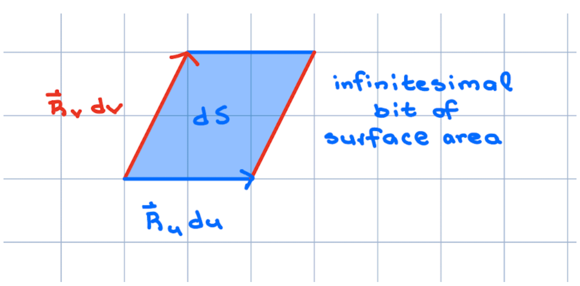
Curves have an orientation, and so too do surfaces. In the picture of $\mathcal{S}$ at the top of the page, the cross product $\vec{R}_u \times \vec{R}_v$ determines upwards normals, which corresponds to an upwards orientation of the surface.

Example 4. Consider cylinder $\mathcal{S}$ given by $x^2 + y^2 = 1$ with $0 \leqslant z \leqslant 2$ and $x, y \geqslant 0$.



Parametrize $\mathcal{S}$, then find a normal to $\mathcal{S}$ in terms of your parameters, and describe the corresponding orientation.

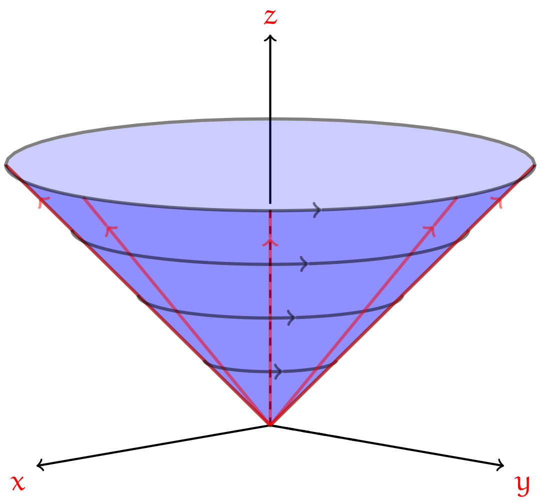
D. **Surface Area.** If we can parametrize a surface $\mathcal{S}$ with parametrization $\vec{R}(u, v)$ with a parameter region D, then its surface area is within reach.



An infinitesimal of surface area can be described using a parallelogram, with sides given by an infitesimal bit of arclength along the u–curve and v–curve respectively. And of course, we cannot forget that one way to compute the area of a parallelogram formed by two vectors is to take the compute the length of their cross–product!

The surface area of a parametrized surface with parameters is:

Example 5. Find the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ with $0 \leqslant z \leqslant 2$.



We will use a pair of cylindrical coordinates to parametrize the cone.

Generally, if we parametrize graph $z = f(r)$ using parameters r and θ, then:

$$dS = r\sqrt{(f'(r))^2 + 1}\, dr d\theta$$