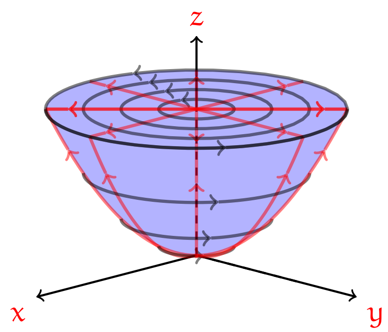
A. **Scalar Surface Integrals.** Just like we can take scalar line integrals by integrating $f\ ds$, we can take scalar surface integrals by integrating $f\ dS$.

The **scalar surface integral** of function $f(x, y, z)$ over $\mathcal{S}$ is:

If the parametrization is $\vec{R}(u, v)$ then here we recall $dS = \|\vec{R}_u \times \vec{R}_v\|\ du dv$.

If $f(x, y, z)$ represents the density of mass per unit area of material that makes up the surface at point (x, y, z), then this integral will compute the mass of the surface.

Example 1. Let $\mathcal{S}$ be the **closed** surface consisting of $z = x^2 + y^2$ with $0 \leqslant z \leqslant 1$ along with its top at $z = 1$.



Let $f(x, y, z) = 1 + z - x^2 - y^2$ and find $\iint_{\mathcal{S}} f \, dS$.

A **closed** surface has an interior that is impossible to enter without penetrating the surface.

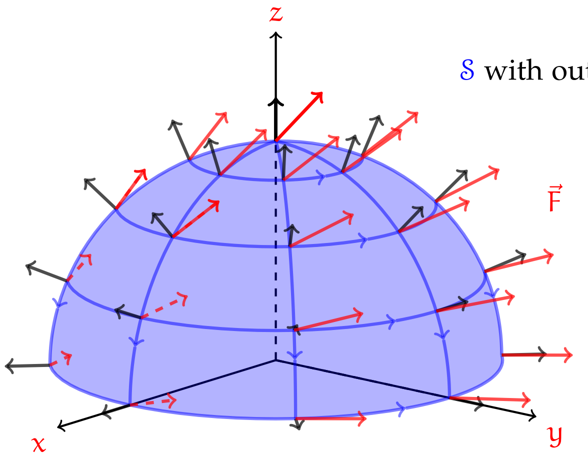
From an earlier margin note, we observed that if we parametrize a graph $z = f(r)$ using r and θ, then:

$dS = r\sqrt{\left[\frac{\partial z}{\partial r}\right]^2 + 1} \, dr d\theta$

Let us use this to save time.

B. **Vector Surface Integrals.** We can also integrate vector fields over surfaces. Imagine an **oriented** surface $\mathcal{S}$ and a fluid velocity field $\vec{F}(x, y, z)$.

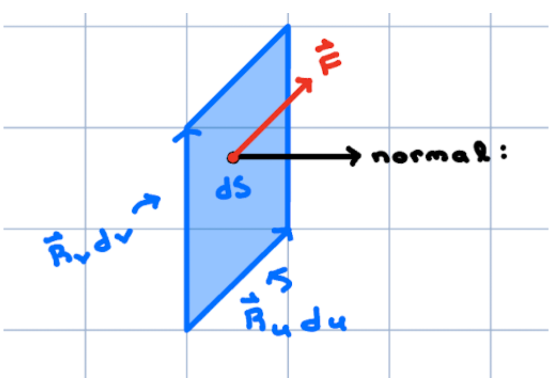
An **oriented** surface is a surface where a choice of orientation, aka a choice of direction for normals, has been selected.



The **flux** of $\vec{F}$ across $\mathcal{S}$ is the rate at which fluid flows across the surface in units of volume per of time: with flow in the direction the orientation counted as positive, and flow in the opposite direction of the oreintation counted as negative.

As you can see in our picture above, sometimes the flow will be in the direction of normals, and sometimes it will be opposite the direction of normals. So some flow will be counted as positive, and some flow will be counted as negative. The "sum" of these contributions is the flux.

Let us compute the flux across an infinitesimal bit of surface area dS, presuming the surface has parametrization $\vec{R}(u, v)$ with parameter region D.



We use $d\vec{S} = \pm(\vec{R}_u \times \vec{R}_v)\, du dv$ to describe the normal vector to the surface with length equal to $\|d\vec{S}\| = dS$, with the choice of $+$ or $-$ made to agree with the orientation.

Remember the component of vector $\vec{w}$ along vector $\vec{v}$ is $\text{comp}_{\vec{v}}(\vec{w}) = (\vec{v} \cdot \vec{w})/\|\vec{v}\|$.

When $d\vec{S}$ is in agreement with the orientation of $\mathcal{S}$, then the flux of $\vec{F}$ across $\mathcal{S}$ is given by the **vector surface integral**:

Example 2. Find the flux of $\vec{F}(x, y, z) = \langle 2x, y, z \rangle$ across the surface $\mathcal{S}$ equal to the portion of the plane by $x + y + z = 1$ in the first octant, and oriented with **downward** normals. First, let's view the situation in Desmos.

Remember the first octant is where $x, y, z \geqslant 0$.

Generally if $z = f(x, y)$ is parametrized using x and y then:

$$d\vec{S} = \pm\langle -\tfrac{\partial z}{\partial x}, -\tfrac{\partial z}{\partial y}, 1\rangle \, dxdy$$

$$dS = \sqrt{\left[\tfrac{\partial z}{\partial x}\right]^2 + \left[\tfrac{\partial z}{\partial y}\right]^2 + 1} \, dxdy$$