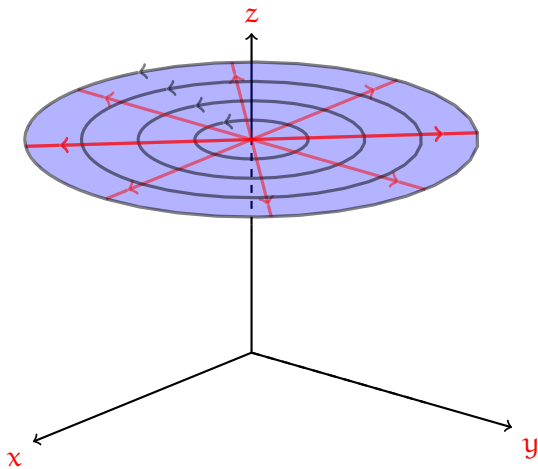


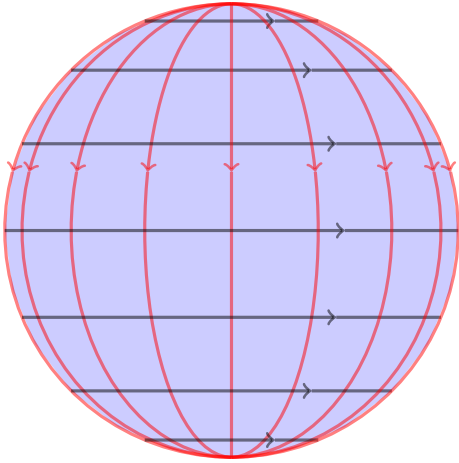
Example 1. \mathcal{S} is the surface $x^2 + y^2 \leq 1$ with $z = 1$ oriented with **upward** normals:



Let \vec{F} be a vector field which, at points in \mathcal{S} , is at angle $\frac{\pi}{3}$ from the upwards normals, and which has length $\|\vec{F}(x, y, z)\| = 3 + z^2$. Find:

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$$

Example 2. Let \mathcal{S} be the sphere $x^2 + y^2 + z^2 = 4$ oriented with **outward normals**:



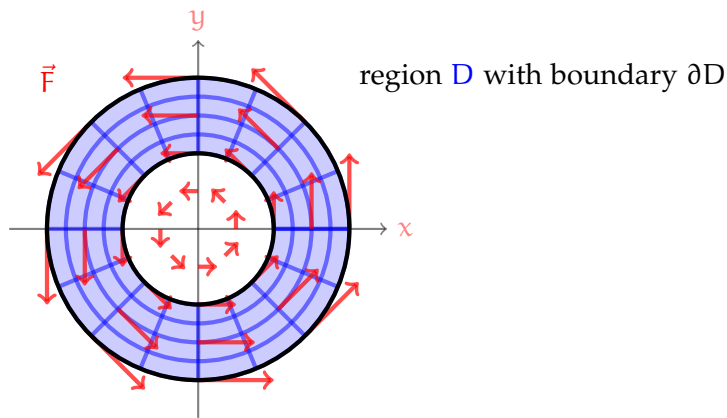
Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and find $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

Generally if the sphere of radius R centered at the origin is parametrized with ϕ and θ then:

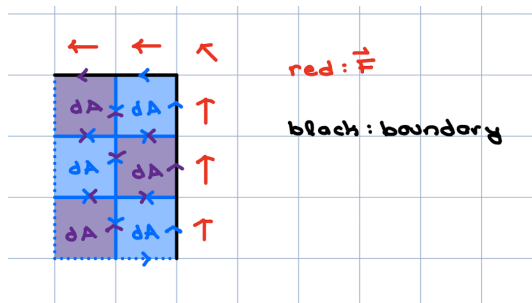
$$d\vec{S} = \pm R \sin \phi \langle x, y, z \rangle d\phi d\theta$$

$$dS = R^2 \sin \phi d\phi d\theta$$

A. **Green’s Theorem.** Our next task is to devise a relationship between line integrals and special kinds of double-integrals. Consider a region D in the xy -plane whose boundary ∂D is made up of **simple**, closed curves, all in the presence of a vector field $\vec{F}(x, y)$.



We divide our region into infinitesimal bits of area, and calculate the circulation about each infinitesimal bit.



Recall that the boundary of a region in the xy -plane consists of those points so that every disk centered at the region both contains points in the region, and points out of the region. The reason for the symbol ∂D to indicate the boundary is tied to a relation between boundaries and derivatives that will be partially revealed when we talk about Stokes’s Theorem.

A closed curve is **simple** if it does **not** cross over itself.



simple closed curve



closed curve, not simple

Recall that curl has to do with circulation. More precisely from an earlier margin note, if we compute the dot product $\text{curl } \vec{F} \cdot \vec{k} \, dA$ then we calculate the circulation about dA , using the righthand rule for out-of-page normals. Here $\vec{k} = \langle 0, 0, 1 \rangle$ is the unit out-of-page vector.

Also remember for a 2D vector field $\vec{F}(x, y) = \langle P, Q \rangle$ that $\text{curl } \vec{F} = (Q_x - P_y)\vec{k}$.

Green’s Theorem. If D is a compact region in the xy -plane whose boundary ∂D is made up of simple, closed curves that have been oriented so that D is on the left, and if $\vec{F}(x, y) = \langle P, Q \rangle$ is defined and continuously differentiable throughout D , then:

The symbol \oint here is an indicator that the curve is closed, nothing more.

Another way to think about the orientation is using the righthand rule with out-of-page normals: situate your righthand along the curve so that your thumb is pointing out-of-the page. If the region is on the left of your hand, then the your hand points in the correct orientation. If the region is on the right of your hand, then your hand points opposite the correct orientation.