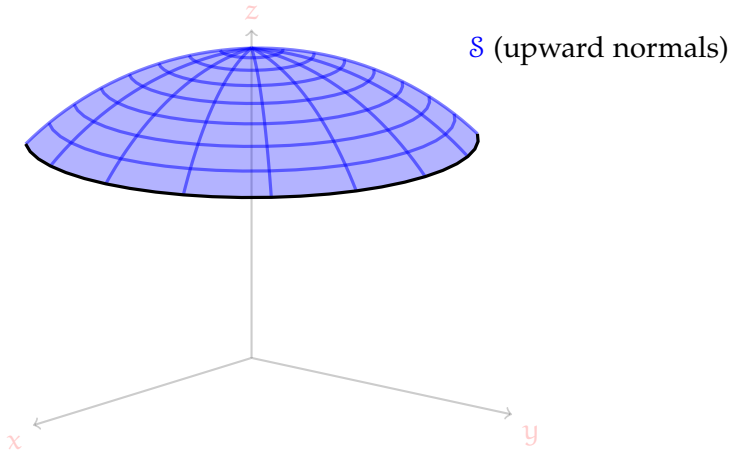


Example 1. Let:

$$\vec{F} = -y\vec{i} + (x + (z-1)xe^x)\vec{j} + (x^2 + y^2)\vec{k}$$

and \mathcal{S} be the spherical cap $x^2 + y^2 + z^2 = 2$ with $z \geq 1$ and oriented with upward normals. Let's view in [Desmos](#).



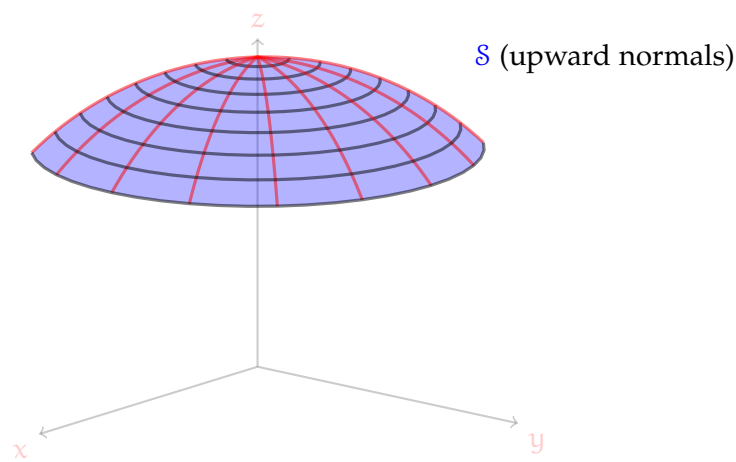
Use Stoke's Theorem to find:

$$\iint_{\mathcal{S}} \text{curl } \vec{F} \cdot d\vec{S} =$$

Example 2. Let’s consider the exact same problem once again:

$$\vec{F} = -y\vec{i} + (x + (z-1)xe^x)\vec{j} + (x^2 + y^2)\vec{k}$$

and \mathcal{S} be the spherical cap $x^2 + y^2 + z^2 = 2$ with $z \geq 1$ and oriented with upward normals.



Recall the shortcut formula for a surface by x and y gives:

$$d\vec{S} = \pm \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle dx dy$$

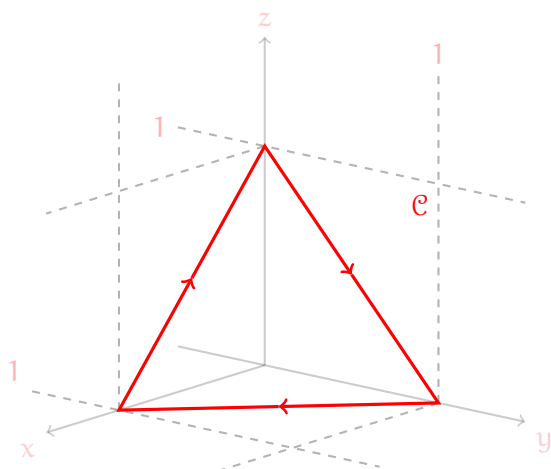
Curls Have Surface-Independent Integrals. If \mathcal{S} and \mathcal{S}' are oriented surfaces with a common boundary, so that the surfaces are on the same side of the boundary when viewed so their normals are pointing at you, then:

The idea is that for curl vector fields, and only for curl vector fields, we can compute a surface integral into an integral over the boundary. In other words, the surface itself does not matter, only its boundary!

Let’s use this to solve the same problem another way. Let’s view in [Desmos](#). Pick a better surface with the same boundary to find:

$$\iint_{\mathcal{S}} \text{curl } \vec{F} \cdot d\vec{S} =$$

Example 3. Let \mathcal{C} be the oriented curve of straight line segments $(1,0,0) \rightarrow (0,0,1) \rightarrow (0,1,0) \rightarrow (1,0,0)$. Let’s view in [Desmos](#).



Use Stokes’s Theorem to find: $\oint_{\mathcal{C}} (x \sin(e^x) - xz) \, dx - xy \, dy + (z^2 + y) \, dz$.

Remember that the symbol \oint just indicates the curve is closed, nothing more.

In this case we find a surface \mathcal{S} having \mathcal{C} as its boundary, and select an orientation for that surface so that \mathcal{S} is on the left of \mathcal{C} , when viewed from a perspective so the normals are pointing at you. Then Stokes’s Theorem let us write:

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \text{curl } \vec{F} \cdot d\vec{S}$$

In this case, the curl of \vec{F} will be significantly nicer than \vec{F} itself, which is why the Stokes’s Theorem approach is recommended.

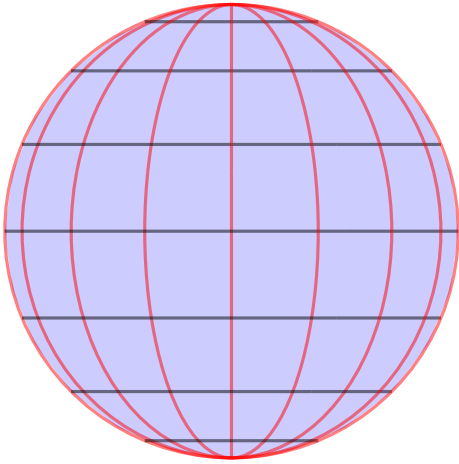
Do not forget that if we parametrize surface using x and y then:

$$d\vec{S} = \pm \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle dx dy$$

Example 4. Earlier we said that a surface is **closed** if it has an interior that was impossible to enter. Now we can be more precise.

A surface is **closed** if it bounded and has **no** boundary curves.

For example the full sphere \mathcal{S} depicted below is a closed surface.



Over a compact, closed surface, the circulations $\text{curl } \vec{F} \cdot d\vec{S}$ about each infinitesimal bit of surface area will all cancel out, because there is no boundary.

If \vec{F} is a vector field that is defined and continuously differentiable throughout a **closed** surface \mathcal{S} , then

$$\oiint_{\mathcal{S}} \text{curl } \vec{F} \cdot d\vec{S} =$$

The \oiint symbol is just an indicator that the surface being integrated over is closed, nothing more.