Math 226 **Final Exam** **Fall 2018**

Last Name: ______________________ **First Name:** ______________________

Signature: ______________________ **Student ID:** ______________________

☐ D. Crombecque 1:00 pm ☐ G. Dreyer 9:00 am ☐ G. Dreyer 10:00 am

☐ N. Haydn 12:00 pm ☐ A. Mazel-Gee 11:00 am ☐ G. Reyes 10:00 am

☐ G. Reyes 1:00 pm ☐ P. Tokorcheck 11:00 pm ☐ P. Tokorcheck 12:00 pm

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Directions. Fill out your name and student ID number. Also check the box next your professor's name/section.

- You must *show all your work and justify your methods* to obtain full credit. Name any theorems that you use. Clearly indicate your final answers.
- You are allowed a double sided HANDWRITTEN sheet of notes on a paper no bigger than $8\frac{1}{2}'' \times 11''$ that should not require any optical device to be read. You may have anything written on it (on both sides), but it must be written in your own handwriting. No other notes or books are allowed during the test.
- No calculators or other electronic devices are allowed. *Turn off your cell phone.*
- Use the back of the pages if additional space is needed.
- Remember, USC considers cheating to be a serious offense; the minimum penalty is failure for the course.

Problem	Points	Score	Problem	Points	Score	Problem	Points	Score
1	20		4	20		7	15	
2	20		5	15		8	20	
3	20		6	20		9	20	
Subtotal	**60**		Subtotal	**55**		Subtotal	**55**	

Total	

Problem 1. Consider the point $A = (3, 1, 4)$ and the line (L) with parametric equations

$$x = 4 - t, \qquad y = 3 + 2t, \qquad z = -5 + 3t$$

a) Prove that A is not on the line (L);

b) Find an equation of the plane containing the line (L) and the point A;

c) Let (K) be the line through the point A that intersects the line (L) orthogonally. Find the coordinates of the intersection point H of the lines (L) and (K). You do not need to simplify each coordinate.

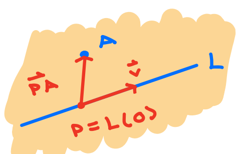
(a)

$A \overset{?}{=} \text{line}(t)$

$$\begin{cases} 3 = 4 - t \longrightarrow t = 1 \\ 1 = 3 + 2t \longrightarrow t = -1 \\ 4 = -5 + 3t \end{cases}$$

] no simultaneous soln

(b)



$P = (4, 3, -5)$

$\vec{v} = \langle -1, 2, 3 \rangle$

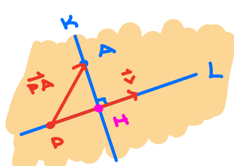
$\vec{PA} = \langle -1, -2, 9 \rangle$

normal: $\vec{v} \times \vec{PA} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 3 \\ -1 & -2 & 9 \end{vmatrix} = \langle 24, 6, 4 \rangle \xrightarrow{\text{rescale}} \vec{n} = \langle 12, 3, 2 \rangle$

plane: $\vec{n} \cdot (\vec{x} - \vec{a}) = 0$

$$12(x-3) + 3(y-1) + 2(z-4) = 0$$

(c)



$H = P + \text{proj}_{\vec{v}} (\vec{PA})$

$= (4,3,-5) + \dfrac{\langle -1,-2,9 \rangle \cdot \langle -1,2,3 \rangle}{\langle -1,2,3 \rangle \cdot \langle -1,2,3 \rangle} \langle -1,2,3 \rangle$

$= (4,3,-5) + \dfrac{\cancel{24}\,12}{\cancel{14}\,7} \langle -1,2,3 \rangle$

$= (4,3,-5) + \langle -\frac{12}{7}, \frac{24}{7}, \frac{36}{7} \rangle$

$$H = \left(\frac{16}{7}, \frac{45}{7}, \frac{1}{7} \right)$$

alt approach: dirn vec for K: $\vec{v}, \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 3 \\ 12 & 3 & 2 \end{vmatrix} = \langle -5, 38, -27 \rangle$

param for K: $\langle 3 - 5s, 1 + 38s, 4 - 27s \rangle$

for L: $\langle 4 - t, 3 + 2t, -5 + 3t \rangle$

$t = 1 + 5s$ $\qquad$ $t = -1 + 19s$

$1 + 5s = -1 + 19s$

$s = \frac{1}{7} \Rightarrow H = \langle 3 - \frac{5}{7}, 1 + \frac{38}{7}, 4 - \frac{27}{7} \rangle$

Problem 2. Consider the two curves C_1 and C_2 respectively parametrized by:

$$\mathbf{r}_1(t) = (t^2+3)\,\mathbf{i} + (t+1)\,\mathbf{j} + \frac{6}{t}\,\mathbf{k}; \qquad t > 0$$

$$\mathbf{r}_2(s) = 4s\,\mathbf{i} + (2s-2)\,\mathbf{j} + (s^2-7)\,\mathbf{k}; \qquad -\infty < s < +\infty$$

a) Find the point of intersection of C_1 and C_2;

b) Find the angle between the curves at the point P found above (do not simplify your answer);

c) Let S be a smooth surface containing both C_1 and C_2. Find an equation of the tangent plane to S at the point P.

(a)
$$\begin{cases} t^2+3 = 4s & (1) \\ t+1 = 2s-2 \longrightarrow t = 2s-3 & (2) \\ \dfrac{6}{t} = s^2-7 & (3) \end{cases}$$

(2) ↳ (1) : $(2s-3)^2 + 3 = 4s$

$4s^2 - 12s + 12 = 4s$

$4s^2 - 16s + 12 = 0$

$4(s^2 - 4s + 3) = 0$

$4(s-3)(s-1) = 0$

$s = 3 \to t = 3$ or $\cancel{s = 1 \to t = -1}$ b/c told $t > 0$

test in (3) : $\frac{6}{3} \stackrel{?}{=} 3^2 - 7$ ✓ $\longrightarrow \vec{r}_1(3) = \langle 12, 4, 3\rangle$ ↲ posn vec for point of intersection

(b) $\vec{r}_1{}'(t) = \langle 2t, 1, -\frac{6}{t^2}\rangle \to \vec{r}_1{}'(3) = \langle 6, 1, -\frac{2}{3}\rangle \xrightarrow{\text{rescale}} \langle 18, 3, -2\rangle = \vec{v}_1$

$\vec{r}_2{}'(s) = \langle 4, 2, 2s\rangle \to \vec{r}_2{}'(3) = \langle 4, 2, 6\rangle \xrightarrow{\text{rescale}} \langle 2, 1, 3\rangle = \vec{v}_2$

$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\|\,\|\vec{v}_2\| \cos\theta$

$33 = \sqrt{337}\,\sqrt{14}\,\cos\theta$

$\cos^{-1}\left(\dfrac{33}{\sqrt{337}\,\sqrt{14}}\right) = \theta$

(c) normal : $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 18 & 3 & -2 \\ 2 & 1 & 3 \end{vmatrix} = \langle 11, -58, 12\rangle$

point : $\vec{r}_1(3) = \vec{r}_2(3) = \langle 12, 4, 2\rangle$

eqn : $11(x-12) - 58(y-4) + 12(z-2) = 0$

topic (implicit diff.) not covered

~~**Problem 3.** Consider the surface S defined by the equation~~

$$\cancel{xz^2 - \arctan(yz) = -\frac{\pi}{4}}$$

a) ~~Find the expressions of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ on S;~~

b) ~~Observe that the point $(x, y, z) = (0, 1, 1)$ lies on S. Using linear approximation (and your solution in part (a)), find an approximation of the z- coordinate of the nearby point on S that has $x = -0.1$ and $y = 1.1$;~~

c) ~~Consider a path $\mathbf{r}(t) = (x(t), y(t), z(t))$ lying in the surface S such that $\mathbf{r}(0) = (0, 1, 1)$. Assume that $\frac{dx}{dt}(0) = -2$ and $\frac{dy}{dt}(0) = 1$. Find the value of $\frac{dz}{dt}(0)$.~~

Problem 4. Consider the following function:

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$$

a) Find the (x,y) - coordinates of all critical points of f in $\mathbb{R}^2$. Classify them as local maximum, local minimum or saddle points.

b) Now consider the closed disk $D = \{(x,y) \mid x^2+y^2 \le 4\}$. Find the (x,y,z) - coordinates of the global (= absolute) extrema of f on D.

(a) $\nabla f = \vec{0}$

$$\begin{cases} 3x^2+6x = 0 \longrightarrow 3x(x+2)=0 \\ 3y^2-6y = 0 \longrightarrow 3y(y-2)=0 \end{cases}$$

crit pts: $(0,0)$, $(0,2)$, $(-2,0)$, $(-2,2)$

$$Hf(x,y) = \begin{pmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{pmatrix}$$

$\det Hf((0,0) \text{ or } (-2,2)) = -36 \rightarrow$ saddle

$\det Hf((0,2) \text{ or } (-2,0)) = 36$

$\operatorname{tr} Hf(0,2) = 12 \rightarrow$ min

$\operatorname{tr} Hf(-2,0) = -12 \rightarrow$ max

(b) candidate in interior: $f(0,0)=0$

edge: Lagrange mult w/ $\underbrace{x^2+y^2}_{g} = 4$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 3x^2+6x = 2\lambda x & ① \\ 3y^2-6y = 2\lambda y & ② \end{cases}$$

Case 1: $x=0 \rightarrow y=\pm 2$. $f(0,2) = -4$, $f(0,-2) = -20$

Case 2: $y=0 \rightarrow x=\pm 2$. $f(2,0) = 20$, $f(-2,0) = 4$

Case 3: $x \neq 0$ and $y \neq 0$

$\frac{①}{x}$: $3x+6 = 2\lambda$

$\frac{②}{y}$: $3y-6 = 2\lambda$

$\longrightarrow 3x+6 = 3y-6 \longrightarrow x+4=y \longrightarrow x^2+(x+4)^2 = 4$

$2x^2+8x+16 = 4$

$2x^2+8x+12=0$

$2(x^2+4x+6)=0$

$x = \dfrac{-4 \pm \sqrt{-24}}{2}$ = no solns

conclusion: max: $f(2,0) = 20$

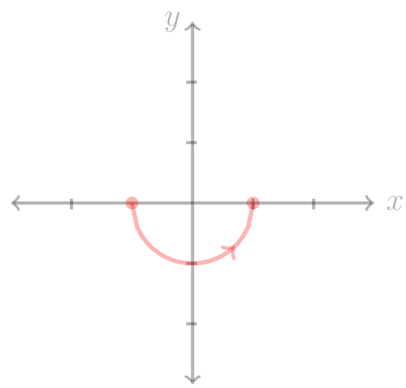
min: $f(0,-2) = -20$

Problem 5. Let $\mathbf{F}$ be the following vector field:

$$\mathbf{F}(x, y) = y^2\,\mathbf{i} + 3xy\,\mathbf{j}.$$

a) Suppose that C_1 is the lower portion of the unit circle $x^2 + y^2 = 1$ going counterclockwise from $(-1, 0)$ to $(1, 0)$.

Compute $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.



$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad \pi \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle \sin^2 t, 3\cos t \sin t \rangle$

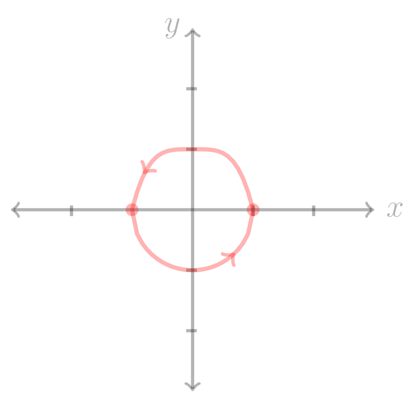
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{\pi}^{2\pi} -\sin^3 t + 3\cos^2 t \sin t \, dt$$

$$= \int_{\pi}^{2\pi} -\sin t\,(1 - \cos^2 t) + 3\cos^2 t \sin t \, dt$$

$$= \int_{\pi}^{2\pi} -\sin t + 4\cos^2 t \sin t \, dt = \left[\cos t - \frac{4}{3}\cos^3 t\right]\Big|_{t=\pi}^{2\pi} = 2 - \frac{8}{3} = -\frac{2}{3}$$

b) Now let the *closed* curve C include the both curve C_1 described above along with the portion of the curve $y = 1 - x^4$ going counterclockwise from $(1, 0)$ to $(-1, 0)$.

Compute $\displaystyle\int_{C} \mathbf{F} \cdot d\mathbf{r}$.



green's : $\displaystyle = \iint_{\text{enclosed}} Q_x - P_y \, dA$

$$= \iint_{\text{enclosed}} y \, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{1-x^4} y \, dy \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - 2x^4 + x^8) - (1 - x^2) \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} x^2 - 2x^4 + x^8 \, dx$$

$$= \frac{1}{2}\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{9}\right)$$

$$= \frac{1}{2}\left(\frac{30 - 36 + 10}{45}\right) = \frac{2}{45}$$

Problem 6. Given the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{ax}{x^2+(y-1)^2}, \frac{y-a}{x^2+(y-1)^2} \right\rangle$$

defined on $\mathbb{R}^2 \setminus \{(0,1)\}$,

a) Find the value(s) of the parameter a for which $\mathbf{F}$ is conservative;

b) For the value of a found in part (a), find a potential for $\mathbf{F}$;

c) For the value of a found in part (a), find the line integral of $\mathbf{F}$ along the circumference

$$(x-1)^2 + y^2 = 1$$

positively oriented;

d) For the value of a found in part (a), find the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

where C is the arc of the parabola $x = y^2$ from $P(1,-1)$ to $Q(4,2)$.

(a) + (b) $\text{curl}\,\vec{F} = \left[\frac{-2x(y-a)}{(x^2+(y-1)^2)^2} - \frac{-2ax(y-a)}{(x^2+(y-1)^2)^2} \right] \vec{k} \overset{\text{set}}{=} \vec{0} \Rightarrow a = 1$

$f(x,y) = \int \frac{x}{x^2+(y-1)^2}\, dx = \frac{1}{2} \ln\left[x^2+(y-1)^2\right] + \cancel{g(y)}$ check ... not needed

(c) $= 0$ b/c curve closed, $\vec{F}$ conservative

(d) $f(4,2) - f(1,-1) = \frac{1}{2}\ln 17 - \frac{1}{2}\ln 5$

Problem 7. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 4$ contained between the planes $z = 1$ and $z = \sqrt{3}$.

spherical:

$$x^2+y^2+z^2 = 4 \rightarrow \rho = 2$$

$$z = 1 \longrightarrow \rho\cos\phi = 1 \longrightarrow 2\cos\phi = 1 \rightarrow \phi = \frac{\pi}{3}$$

$$z = \sqrt{3} \longrightarrow \rho\cos\phi = \sqrt{3} \longrightarrow 2\cos\phi = \sqrt{3} \rightarrow \phi = \frac{\pi}{6}$$

parametrize:

$$\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle \qquad 0 \le \theta \le 2\pi, \quad \frac{\pi}{6} \le \phi \le \frac{\pi}{3}$$

shortcut ($\rho = R$)
$\vec{r}_\phi \times \vec{r}_\theta = R\sin\phi\, \vec{r}(\phi,\theta)$

$$\vec{r}_\phi \times \vec{r}_\theta = 2\sin\phi\, \vec{r}(\phi,\theta)$$

$$\|\vec{r}_\phi \times \vec{r}_\theta\| = 2\sin\phi\, \|\vec{r}(\phi,\theta)\| = 2\sin\phi \cdot 2 = 4\sin\phi$$

so: $$\int_0^{2\pi}\int_{\pi/6}^{\pi/3} 4\sin\phi \, d\phi\, d\theta = 2\pi \cdot 4 \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$= 4\pi(\sqrt{3}-1)$$

Problem 8. Evaluate the integral

$$\int_C y\,dx + x^2\,dy + xz^3\,dz$$

where C is the triangle with vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$, oriented clockwise if seen from above.

$\mathcal{C}$ boundary of $\mathcal{S}$: $x+y+z=2$ with $x,y,z \geq 0$

$\mathcal{C}$ $\downarrow$ clockwise from above $\longrightarrow$ $\mathcal{S}$ $\downarrow$ downward

parametrize : $\vec{r}(x,y) = \langle x, y, 2-x-y \rangle$ $\quad 0 \leq y \leq 2-x$, $\ 0 \leq x \leq 2$

$\vec{r}_x \times \vec{r}_y = \langle 1,1,1 \rangle$

$\downarrow$ downward

$\langle -1,-1,-1 \rangle$

$\vec{F} = \langle y, x^2, xz^3 \rangle$

$\text{curl}\,\vec{F} = \langle 0, -z^3, 2x-1 \rangle$

$\text{curl}\,\vec{F}(\vec{r}(x,y)) = \langle 0, -(2-x-y)^3, 2x-1 \rangle$

Stokes : $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot d\vec{s} = \int_0^2\int_0^{2-x} \langle 0, -(2-x-y)^3, 2x-1 \rangle \cdot \langle -1,-1,-1 \rangle \, dy\,dx$

$\displaystyle = \int_0^2\int_0^{2-x} (2-x-y)^3 + 1 - 2x \; dy\,dx$

$\displaystyle = \int_0^2 \left[-\frac{1}{4}(2-x-y)^4 + (1-2x)y \right]\Big|_{y=0}^{2-x} dx$

$\displaystyle = \int_0^2 \frac{1}{4}(2-x)^4 + (1-2x)(2-x)\,dx$

$\displaystyle = \int_0^2 \frac{1}{4}(2-x)^4 + 2 - 5x + 2x^2\,dx$

$\displaystyle = -\frac{1}{20}(2-x)^5 + 2x - \frac{5}{2}x^2 + \frac{2}{3}x^3 \Big|_{x=0}^{2}$

$\displaystyle = \frac{\cancel{32}^{\,8}}{\cancel{20}_{\,5}} + 4 - 10 + \frac{16}{3}$

$\displaystyle = \frac{24}{15} - \frac{90}{15} + \frac{80}{15}$

$\displaystyle = \frac{14}{15}$

Problem 9. Let E be the part of the solid the cylinder $x^2 + y^2 \le 4$ in the first octant and bounded by the plane $z = 3$. We call S the boundary of E, oriented outwards. Consider the vector field

$$\mathbf{F} = (6x^2 + 2xy)\,\mathbf{i} + (2y + x^2z)\,\mathbf{j} + 4x^2y^3\,\mathbf{k}.$$

Use the method of your choice to compute the flux of the vector field $\mathbf{F}$ out of S.

first octant

E in cylindrical : $0 \le z \le 3$

$0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

first octant

$\operatorname{div} \vec{F} = 12x + 2y + 2$

$$\underset{\substack{S \\ \text{outward}}}{\iint} \vec{F} \cdot d\vec{S} = \underset{E}{\iiint} 12x + 2y + 2 \, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^3 12r^2\cos\theta + 2r^2\sin\theta + 2r \, dz\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^2 36r^2\cos\theta + 6r^2\sin\theta + 6r \, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 96\cos\theta + 16\sin\theta + 12 \, d\theta$$

$$= 96 + 16 + 6\pi$$

$$= 112 + 6\pi$$