Math 226 Final Exam
Fa25
Wed Dec 10

Firstname Lastname: ______________________________

USC Email: ______________________ UscID: ______________________

Instructions.

- This examination consists of 20 pages not including this cover page.
- Write your initials in the designated spot at the top–right of each page that contains work which you would like to have graded.
- This examination consists of 9 questions for a total of 100 points. You have 120 minutes to complete this examination.
- Do not use books, calculators, computers, tablets, or phones.
- You may use a single 8.5 in by 11 in page of notes, handwritten on both sides.
- Write legibly in the boxed area only. Cross out any work that you do not wish to have scored.
- Show all of your work and cite theorems you use. Unsupported answers may not earn credit.
- If you run out of space: there is an extra page after each problem, and one extra page at the end. Please indicate on the page containing the original question when you are continuing your work on another page.
- All work you submit should represent your own thoughts and ideas. If the graders suspect otherwise: you can expect your instructor to file a report with USCs Office of Academic Integrity (OAI).

Circle your instructor: Prof Geske Prof Haine Prof Reyes Souto Prof Ziane

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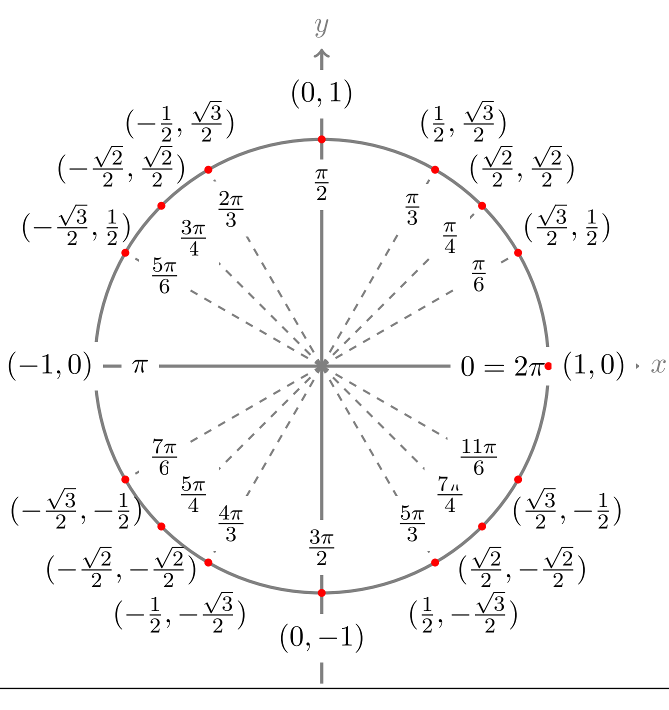
Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	12	10	10	10	10	12	12	14	100



$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

1. (10 points) Consider the parametric curve $\mathbf{r}(t) = \langle \sin(t), \cos(3t), 4t^2 \rangle$ with domain $(-\infty, \infty)$.

   (a) Give a parametrization of the line L tangent to this curve at the point $P(1, 0, \pi^2)$.

Solution: We find the parameter t yielding P by solving:

$$\begin{cases} \sin(t) = 1 \\ \cos(3t) = 0 \\ 4t^2 = \pi^2 \end{cases}$$

The last equation tell us $t = \pm\frac{\pi}{2}$. In order to satisfy the first equation, $\sin(t) = 1$, of the $\pm$, we select $t = \frac{\pi}{2}$. This also satisfies the second equation. So: $P = \mathbf{r}(\frac{\pi}{2})$.
Next, we calculate:

$$\mathbf{r}'\left(\frac{\pi}{2}\right) = \left\langle \cos\left(\frac{\pi}{2}\right), -3\sin\left(\frac{3\pi}{2}\right), 8 \cdot \frac{\pi}{2} \right\rangle = \langle 0, 3, 4\pi \rangle$$

The parametrization is thus:

$$\ell(s) = \mathbf{p} + s\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 1, 3s, \pi^2 + 4\pi s \rangle$$

   (b) Write down the Cartesian equation (aka scalar equation) for the plane passing through P (from part a) and orthogonal to L (also from part a).

Solution: The normal is $\mathbf{n} = \mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 0, 3, 4\pi \rangle$. So the scalar equation is:

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \implies 3y + 4\pi z = 4\pi^3$$

You may continue your work for Q1 on this page.

2. (12 points) Let $f(x, y) = 2x - xy - xy^2 + y^3$.

(a) Find, and classify, all critical points of $f(x, y)$.

Solution: We set the partial derivatives equal to 0:

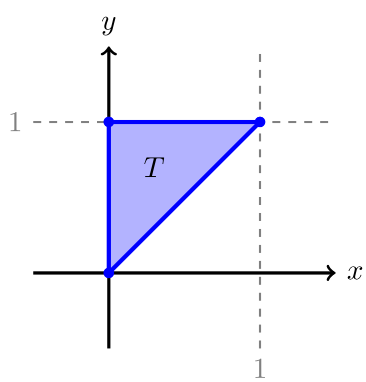
$$\begin{cases} 2 - y - y^2 = 0 \implies 0 = y^2 + y - 2 \implies 0 = (y+2)(y-1) \\ -x - 2xy + 3y^2 = 0 \end{cases}$$

The first equation gives us $y = -2$ or $y = 1$.
If $y = -2$ then the second equation reads $-x + 4x + 12 = 0 \implies x = -4$. So one critical point is $(-4, -2)$.
If $y = 1$ then the second equation reads $-x - 2x + 3 = 0 \implies x = 1$. So the other critical point is $(1, 1)$.
The Hessian is:

$$Hf(x, y) = \begin{pmatrix} 0 & -1 - 2y \\ -1 - 2y & \star \end{pmatrix}$$

Thus, the Hessian determinant is $\det Hf(x, y) = -(1 + 2y)^2$ which is always negative. So, all critical points are saddles points are saddles.

(b) Find the absolute maximum and minimum values of f on the closed triangular region T depicted below.



Solution: There are no critical points in the interior. Only the edges need be considered.
Along the edge $x = 0$, with $0 \le y \le 1$, we have:

$$f(0, y) = y^3$$

which is maximized at $f(0, 1) = 1$ and minimized at $f(0, 0) = 0$.
Along the edge $y = 1$, with $0 \le x \le 1$, we have:

$$f(x, 1) = 2x - x - x + 1 = 1$$

which is constant at value 1.
Along the edge $y = x$, with $0 \le x \le 1$, we have:

$$f(x, x) = 2x - x^2 - x^3 + x^3 = 2x - x^2 \implies \frac{d}{dx}\Big[f(x, x)\Big] = 2 - 2x$$

and so $f(x, x)$ has critical point at $x = 1$. We test critical points and endpoints: $f(0, 0) = 0$ and $f(1, 1) = 1$.
Therefore, f has maximum value $f = 1$, achieved along top edge of the triangle, and has minimum value $f = 0$, achieved at $(0, 0)$.

You may continue your work for Q2 on this page.

3. (10 points) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x^2y + 1$ subject to $2x^2 + 4y^2 = 3$.

Solution: We set up the Lagrange equations:

$$\begin{cases} 2xy = 4x\lambda \implies 2x(y - 2\lambda) = 0 \\ x^2 = 8y\lambda \end{cases}$$

The first equation yields either $x = 0$ or $y = 2\lambda$.

If $x = 0$, substituting into the constraint gives:

$$4y^2 = 3 \implies y = \pm\frac{\sqrt{3}}{2}$$

thus giving us Lagrange solutions:

$$f\left(0, \pm\frac{\sqrt{3}}{2}\right) = 1$$

If $y = 2\lambda \implies \lambda = \frac{1}{2}y$, then substituting into the second equation gives:

$$x^2 = 4y^2$$

and then writing the constraint in y gives:

$$8y^2 + 4y^2 = 3 \implies y^2 = \frac{1}{4} \implies y = \pm\frac{1}{2}$$

Altogether, this gives us the Lagrange solutions:

$$f\left(\pm 1, \frac{1}{2}\right) = \frac{3}{2} \text{ and } f\left(\pm 1, -\frac{1}{2}\right) = \frac{1}{2}$$

Comparing values, the maximum value of is $f = \frac{3}{2}$, and the minimum value is $f = \frac{1}{2}$.

You may continue your work for Q3 on this page.

4. (10 points) Consider the solid E bounded by the surface $y^2 = x(2-z)$ and the plane $x + z = 2$, and contained in the first octant, $x, y, z \geq 0$. Assume x, y, z are measured in cm. The density of the solid E at point (x, y, z) is given by $\delta(x, y, z) = 2y\ g/\text{cm}^3$. Calculate the total mass of E.

   Hint: an integral in order $dyd\star d\star$ may be most appropriate.

Solution: Let E be this region. The shadow of this region in the xz-plane is the triangle:

$$\begin{cases} 0 \leq z \leq 2 - x \\ 0 \leq x \leq 2 \end{cases}$$

Importantly, in this range, $x(2-z)$ is always nonegative. So, in the first octant:

$$y^2 = x(2-z) \implies y = \sqrt{x(2-z)}$$

For each (x, z) from this shadow, the range of y is thus:

$$0 \leq y \leq \sqrt{x(2-z)}$$

We are now ready to set up our integral.

$$\iiint_E \delta(x, y, z)\ dV = \int_0^2 \int_0^{2-x} \int_0^{\sqrt{x(2-z)}} 2y\ dydzdx = \int_0^2 \int_0^{2-x} x(2-z)\ dzdx = \cdots$$

$$\cdots = \frac{1}{2} \int_0^2 4x - x^3\ dx = \frac{1}{2}(8-4) = 2$$

So the mass of E is $2\,\text{g}$.

You may continue your work for Q4 on this page.

5. (10 points) Consider the region R defined by the inequalities:

$$x^2 + y^2 + z^2 \leq 9 \quad \text{and} \quad -\frac{1}{\sqrt{3}}\sqrt{x^2+y^2} \leq z \leq 0$$

Compute the value of the triple integral:

$$\iiint_R \frac{\arcsin\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)}{\sqrt{x^2+y^2}}\, dV$$

Hint: recall that, if $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, then $\arcsin(\sin t) = t$. Recall also that $\sin(t) = \sin(\pi - t)$.

Hint: this integral may look scary as presented, but in spherical coordinates looks far less scary.

Solution: In spherical the region is:

$$\begin{cases} 0 \leq \rho \leq 3 \\ \frac{\pi}{2} \leq \phi \leq \frac{2\pi}{3} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

We compute:

$$\int_0^{2\pi} \int_{\pi/2}^{2\pi/3} \int_0^3 \frac{\arcsin(\sin\phi)}{\rho\sin\phi}\,\rho^2 \sin\phi d\rho d\phi d\theta = \cdots$$

Next, notice that ϕ is in the range $[\frac{\pi}{2}, \pi]$, in which case $\pi - \phi$ is in the range $[0, \frac{\pi}{2}]$, and thus:

$$\arcsin(\sin\phi) = \arcsin(\sin(\pi - \phi)) = \pi - \phi$$

so that we may conntinue:

$$\cdots = \int_0^{2\pi} \int_{\pi/2}^{2\pi/3} \int_0^3 \rho \cdot (\pi - \phi) d\rho d\phi d\theta = \cdots$$

$$\cdots = 2\pi \cdot \frac{9}{2} \cdot \left(\frac{\pi^2}{6} - \frac{7\pi^2}{72}\right) = 2\pi \cdot \frac{9}{2} \cdot \left(\frac{5\pi^2}{72}\right) = \frac{5\pi^3}{8}$$

You may continue your work for Q5 on this page.

6. (10 points) Consider the vector force field, with domain the half–plane $x + y > 0$, given by:

$$\mathbf{F}(x,y) = \left[ye^x + \frac{y}{x+y}\right]\mathbf{i} + \left[2 + e^x + \frac{y}{x+y} + A\ln(x+y)\right]\mathbf{j}$$

(a) There is a value of constant A so that $\mathbf{F}$ is conservative on its domain. Find that value.

Solution: Note that the domain is simply–connected, and therefore to verify that $\mathbf{F}$ is conservative, we need only confirm that it is irrotational. To do this we set up the equality:

$$\frac{\partial}{\partial y}\left[ye^x + \frac{y}{x+y}\right] = \frac{\partial}{\partial x}\left[2 + e^x + \frac{x}{x+y} + A\ln(x+y)\right] \implies e^x + \frac{x}{(x+y)^2} = e^x - \frac{y}{(x+y)^2} + \frac{A}{x+y}$$

$$\implies \frac{x+y}{(x+y)^2} = \frac{A}{x+y} \implies \frac{1}{x+y} = \frac{A}{x+y} \implies A = 1$$

(b) Using the value of A you found in part (a), find a potential f for $\mathbf{F}$.

Solution: Now, suppose that $A = 1$. We look for a potential f. We have:

$$f(x,y) = \int ye^x + \frac{y}{x+y}\,dx = ye^x + y\ln(x+y) + g(y)$$

Next we take the y-derivative of our expression, and equate it to the second component of $\mathbf{F}$, again assuming $A = 1$, to get:

$$e^x + \ln(x+y) + \frac{y}{x+y} + g'(y) = 2 + e^x + \frac{y}{x+y} + \ln(x+y) \implies g'(y) = 2 \implies g(y) = 2y + C$$

So, the general potential function has form:

$$f(x,y) = ye^x + y\ln(x+y) + 2y + C$$

Note: another equally valid approach to this problem, is to begin by attempting to find a single potential function, and then noting that if $A = 1$, this task is feasible, and lastly obtaining the potential, as above.

(c) Using the value of A you found in part (a), evaluate the work done by the conservative force field $\mathbf{F}$ on a particle moving along the path:

$\mathbf{r}(t) = \left\langle \cos^4 t, \frac{2t}{\pi}\right\rangle$ with $0 \le t \le \pi$

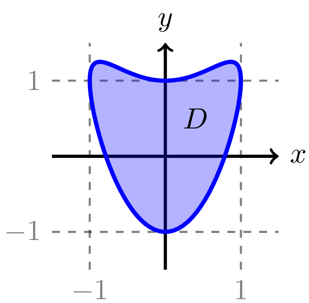
Solution: The initial value of our path is $\mathbf{r}(0) = \langle 1, 0\rangle$, and the terminal value of our path is $\mathbf{r}(\pi) = \langle 1, 2\rangle$. By the fundamental theorem of line-integrals, the work is equal to:

$$f(1,2) - f(1,0) = 2e + 2\ln(3) + 4$$

You may continue your work for Q6 on this page.

7. (12 points) The path:

$\mathbf{r}(t) = \left\langle \cos t, \cos^2 t + \sin t \right\rangle$ with $0 \le t \le 2\pi$ encloses the following region D.



(a) Use Green's Theorem, by computing an appropriate line integral, to find the area of the region D.

Hint: there are trig identities on the cover page.

Solution: According to Green's Theorem:

$$\int_{\mathbf{r}} -y \, dx = \iint_D 1 \, dA = \text{area}(D)$$

So we compute:

$$dx = x'(t) \, dt = -\sin t \, dt$$

and thus:

$$\int_{\mathbf{r}} -y \, dx = \int_0^{2\pi} -(\cos^2 t + \sin t)(-\sin t) \, dt = \int_0^{2\pi} \sin t \cos^2 t + \sin^2 t \, dt = \pi$$

(b) Find:

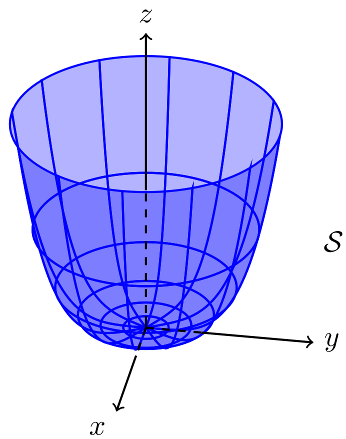
$$\oint_{\mathbf{r}} \left[e^{x^2} + y^2 + 2y \right] \, dx + \left[2xy + \sin\left(y^2\right) \right] \, dy$$

Note: if you would like to use the answer to part (a) for part (b), but were unable to solve part (a), then use symbol A to represent the answer to part (a).

Solution: Using Green's Theorem, equals $\iint 2y - 2y - 2 \, dA = \iint -2 \, dA = -2\pi$.

You may continue your work for Q7 on this page.

8. (12 points) Consider the surface $\mathcal{S}$ equal to the portion of $z = (x^2+y^2)^2$ with $z \leq 1$. Orient $\mathcal{S}$ with upwards normals.



$\mathcal{S}$ (like an upwards bowl)

Consider also the vector field:

$\mathbf{F}(x, y, z) = \langle z, x, \sin^2(z-1)\rangle$

Confirm Stokes's Theorem for this $\mathcal{S}$ and $\mathbf{F}$ by computing a relevant surface integral and a relevant line integral, and confirming equality. This is divided into parts below.

(a) For this part, directly compute the **surface** integral relevant to Stokes's Theorem. Note: you must use **surface** integral, not line integral, calculations here.

Hint: one approach is to use polar coordinates as parameters for $\mathcal{S}$. Be careful, you are not computing the surface integral of $\mathbf{F}$ here, rather, you are computing the surface integral of ......

Solution: The surface in question can be written in cylindrical coordinates as:

$$z = (r^2)^2 \implies z = r^4 \text{ where } 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi$$

We calculate:

$$\operatorname{curl} \mathbf{F} = \langle 0, 1, 1\rangle$$

The shortcut for upwards $d\mathbf{S}$ for a graph $z = f(r)$ is:

$$d\mathbf{S} = \langle -f'(r) r\cos\theta, -f'(r) r\sin\theta, r\rangle$$

which, since $f(r) = r^4$ in our case, yields:

$$d\mathbf{S} = \langle -4r^4\cos\theta, -4r^4\sin\theta, r\rangle \; drd\theta$$

So:

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 -4r^4\sin\theta + r \; drd\theta = \pi$$

Note: The question is ambiguous, and therefore if you opt to use the method of changing the surface, for example by replacing $\mathcal{S}$ with a nicer $\mathcal{S}'$, like a flat disk, then you still have the possibility of earning full credit. The key is: a correct solution should be using relevant **surface** integral calculations.

(b) For this part, directly compute the **line** integral relevant to Stokes's Theorem. Confirm that your answers to the two parts are equal. Note: you must use **line** integral, not surface integral calculations here.

Solution: The boundary is along the top edge, $z = 1$, which, when plugged in to the surface equation, yields:

$$1 = (x^2 + y^2)^2 \implies 1 = r^4 \implies r = 1$$

In order to be compatible with the upward normals, the boundary should be oriented counterclockwise when viewed from above. The standard parametrization of a circle using the polar θ is counterclockwise, so we use it to get:

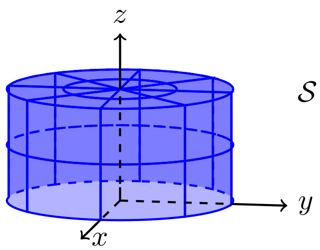
$$\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, 1\rangle \implies d\mathbf{r} = \langle -\sin\theta, \cos\theta, 0\rangle\ d\theta$$

and then we integrate:

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -\sin t + \cos^2 t\ dt = \pi$$

You may continue your work for Q8 on this page.

9. (14 points) Let $\mathcal{S}$ consist of the cylinder defined by $x^2 + y^2 = 1$ with $0 \le z \le 1$ along with its **top** $x^2 + y^2 \le 1$ with $z = 1$. Orient $\mathcal{S}$ with outward normals.



$\mathcal{S}$ (top included, bottom not included)

(a) The surface $\mathcal{S}$ is not closed. Describe the simplest surface $\mathcal{B}$ that you would have to join to $\mathcal{S}$ to obtain a closed surface. To be compatible with outward normals for the closed surface: should the normals to $\mathcal{B}$ point up or down?

Solution: The bottom $\mathcal{B} = \{x^2 + y^2 \le 1 \text{ and } z = 0\}$. Oriented downwards.

(b) Find $\iint_{\mathcal{B}} \mathbf{F} \cdot d\mathbf{S}$ where $\mathcal{B}$ is from part (a) and:

$$\mathbf{F} = \left(x + ye^{z^3}\right)\mathbf{i} + \left(z^2\cos(x^2) - y\right)\mathbf{j} + \left(1 + z\right)\mathbf{k}$$

Solution: Because $\mathcal{B}$ is flat, and oriented with downards normals, it follows that $d\mathbf{S} = \langle 0, 0, -1\rangle\, dA$. So:

$$\iint_{\mathcal{B}} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2\le 1} \langle \star, \star, 1+0\rangle \cdot \langle 0,0,-1\rangle\, dA = \iint_{x^2+y^2\le 1} -1\, dA = -\text{area}(x^2+y^2 \le 1) = -\pi$$

(c) Using a combination of the divergence theorem and your earlier work, find:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}$ is from part (b) and $\mathcal{S}$ is from the top of the page.

Solution: Let E be the solid cylinder $x^2 + y^2 \le 1$ with $0 \le z \le 1$. By the divergence theorem, since $\mathcal{S}$ and $\mathcal{B}$ together form the boundary of E, with outward normals, we have:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{B}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\text{E}} \text{div}\,\mathbf{F}\, dV$$

Next we calculate:

$$\text{div}\,\mathbf{F} = 1 - 1 + 1 = 1$$

and:

$$\iiint_{\text{E}} \text{div}\,\mathbf{F}\, dV = \iiint_{\text{E}} 1\, dV = \text{volume(E)} = \pi$$

Returning to the divergence theorm calculation, and substituting discovered values:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \pi = \pi \implies \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = 2\pi$$

You may continue your work for Q9 on this page.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.