MATH 226: Final Exam

Saturday, May 4th, 2024

Full name: ________________________

USC ID: ________________________

Signature: ________________________

INSTRUCTIONS:

- Fill out your full name, student ID number, and signature.
- You must show all of your work, and carefully explain your methods and reasoning. You must name any theorems or tests that you use. Solutions must be neat, legible, well organized, and unambiguous. Final answers must be optimally simplified and clearly indicated.
- You are allowed a half of a regular $8.5'' \times 11''$ page, 1–sided, HANDWRITTEN formula sheet that should not require any optical device to be read. You must include your full name on it. Your sheet will be collected at the end of the examination. No other notes or books are allowed during the test.
- No calculators, smart watches, or other electronic devices are allowed. Turn off your cell phone. No communication, no collaboration with anyone, no lurking eyes. Failure to comply with the above instructions will be deemed a violation of university policy; the minimum penalty for academic dishonesty is failure in the course.

Problem 1. ($\approx$ 18 points) In all of the following questions, you must clearly explain your methods and justify your reasoning.

(a) Find a parametrization of the line $\mathcal{L}$ which intersects the planes $x - 2y + 3z + 7 = 0$ and $-x + 3y - z - 4 = 0$.

$\downarrow$ $\vec{n}_1 = \langle 1, -2, 3 \rangle$

$\downarrow$ $\vec{n}_2 = \langle -1, 3, -1 \rangle$

dirn vec: $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ -1 & 3 & -1 \end{vmatrix} = \langle -7, -2, 1 \rangle$

point on line w/ $z = 0$: $\begin{cases} x - 2y + 7 = 0 \rightarrow x = 2y - 7 \quad ① \\ -x + 3y - 4 = 0 \quad ② \end{cases}$

① $\rightarrow$ ②: $7 - 2y + 3y - 4 = 0 \rightarrow y = -3 \rightarrow x = -13$

parametrize: $\langle -13 - 7t, -3 - 2t, t \rangle$

(b) Find an equation of the plane $\mathcal{P}$ parallel to the lines $\mathbf{r}_1(t) = \langle -2t + 1, 2 - t, 3t - 1 \rangle$ and $\mathbf{r}_2(t) = \langle t + 5, 3t - 2, 4t + 3 \rangle$ and containing the point $(-2, 3, -4)$.

normal: $\langle -2, -1, 3 \rangle \times \langle 1, 3, 4 \rangle = \langle -13, 11, -5 \rangle$

plane: $-13(x + 2) + 11(y - 3) - 5(z + 4) = 0$

(c) Determine the distance between the line $\mathbf{r}_3(t) = \langle 7t + 1, 2t + 1, 3 - t \rangle$ and the plane $x - 2y + 3z + 7 = 0$.

is $\vec{r}_3$ parallel to plane?

normal · dirn $\stackrel{?}{=} 0$

$\langle 1, -2, 3 \rangle \cdot \langle 7, 2, -1 \rangle = 7 - 4 - 3 \stackrel{\checkmark}{=} 0$

yes!

next: point on line : $Q\ (1, 1, 3)$

point on plane : $P\ (-7, 0, 0)$

$\vec{PQ} = \langle 8, 1, 3 \rangle$

distance : $\left| \text{comp}_{\text{normal}} (\vec{PQ}) \right| = \dfrac{|\langle 1, -2, 3 \rangle \cdot \langle 8, 1, 3 \rangle|}{\| \langle 1, -2, 3 \rangle \|} = \dfrac{15}{\sqrt{14}}$

Problem 2. ($\approx$ 17 points) Suppose $z = f(x, y)$, where:

$$f(1,3) = 7$$
$$f_x(1,3) = 5$$
$$f_y(1,3) = -4.$$

(a) Find the directional derivative of f at $(1,3)$ in the direction of $\mathbf{u} = \langle \frac{4}{5}, \frac{3}{5} \rangle$.

$$D_{\vec{u}} f(1,3) = \nabla f(1,3) \cdot \vec{u}$$
$$= \langle 5, -4 \rangle \cdot \langle \tfrac{4}{5}, \tfrac{3}{5} \rangle$$
$$= 4 - \frac{12}{5}$$
$$= \frac{18}{5}$$

(b) Furthermore, assume that $x = g(s,t)$, $y = 2s + t$, and:

$$x(1,1) = g(1,1) = 1 \quad , \quad y(1,1) = 3$$
$$C = \left.\frac{\partial x}{\partial s}\right|_{(1,1)} = \left.\frac{\partial x}{\partial t}\right|_{(1,1)} \quad , \quad y_s(1,1) = 2, \quad y_t(1,1) = 1$$
$$\left.\frac{\partial z}{\partial s}\right|_{(1,1)} = 2.$$

f same from top of page

Find $\frac{\partial \boxed{z}}{\partial t}$ at $(s,t) = (1,1)$.

$$① : z_s(\overset{s}{1}, \overset{t}{1}) = z_x(\overset{x}{1}, \overset{y}{3}) \cdot x_s(\overset{s}{1}, \overset{t}{1}) + z_y(\overset{x}{1}, \overset{y}{3}) \cdot y_s(\overset{s}{1}, \overset{t}{1})$$

$$② : z_t(\overset{s}{1}, \overset{t}{1}) = z_x(\overset{x}{1}, \overset{y}{3}) \cdot x_t(\overset{s}{1}, \overset{t}{1}) + z_y(\overset{x}{1}, \overset{y}{3}) \cdot y_t(\overset{s}{1}, \overset{t}{1})$$

sub in...

$$① : 2 = 5C - 8$$
$$② : z_t(1,1) = 5C - 4$$

$$② - ① : z_t(1,1) - 2 = 4$$
$$z_t(1,1) = 6$$

(c) Using linear approximation, find an approximate value of z when $s = 0.9$ and $t = 1.1$.

$$z(s,t) \approx z(\overset{s}{1},\overset{t}{1}) + z_s(\overset{s}{1},\overset{t}{1}) \cdot (s-1) + z_t(\overset{s}{1},\overset{t}{1}) \cdot (t-1)$$

$$z(s,t) \approx 7 + 2(s-1) + 6(t-1)$$

$$z(\overset{s}{0.9},\overset{t}{1.1}) \approx 7 - 0.2 + 0.6 = 7.4$$

Problem 3. (≈ 20 points) Consider the function $f(x,y) = y^3 + 3x^2y + 2$ on the domain $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$. Show that f achieves absolute extrema on $\mathcal{D}$, then determine the values of these extrema. You must carefully explain your methods, and optimally simplify your answers.

D :  bounded + closed $\Rightarrow$ compact

f is continuous (as a polynomial)

so: by EVT f achieves extremizers on D

crit pts:

$\nabla f = \vec{0}$

$$\begin{cases} 6xy = 0 \\ 3y^2 + 3x^2 = 0 \end{cases} \Rightarrow x = 0, y = 0$$

candidate: $f(0,0) = 2$

edge: Lagrange multipliers, $x^2 + y^2 = 4$, $g = 4$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 6xy = 2x\lambda & (1) \\ 3y^2 + 3x^2 = 2y\lambda & (2) \end{cases}$$

Case 1: $x = 0 \rightarrow y = \pm 2$. $f(0,2) = 10$, $f(0,-2) = -6$

Case 2: $x \neq 0$

$\frac{(1)}{2x}$: $3y = \lambda$

$\hookrightarrow$ (2): $3y^2 + 3x^2 = 6y^2$

$x^2 = y^2$

$\hookrightarrow g = 4$: $x^2 + x^2 = 4 \rightarrow x = \pm\sqrt{2}$

candidates: $f(\pm\sqrt{2}, \sqrt{2}) = 2\sqrt{2} + 6\sqrt{2} + 2 = 8\sqrt{2} + 2$

$f(\pm\sqrt{2}, -\sqrt{2}) = -2\sqrt{2} - 6\sqrt{2} + 2 = 2 - 8\sqrt{2}$

conclusion: max $f(\pm\sqrt{2}, \sqrt{2}) = 8\sqrt{2} + 2$

min $f(\pm\sqrt{2}, -\sqrt{2}) = 2 - 8\sqrt{2}$

Problem 4. (≈ 17 points) Let C_1 and C_2 be the curves described by the vector functions:

$$\mathbf{r}_1(t) = \langle\, 2t+1, t, t^2-1 \,\rangle$$
$$\mathbf{r}_2(s) = \langle\, 2s^3+3, s^2+1, 3s \,\rangle.$$

(a) Find all points of intersection of C_1 and C_2.

$$\begin{cases} 2t+1 = 2s^3+3 & (1) \\ t = s^2+1 & (2) \\ t^2-1 = 3s & (3) \end{cases}$$

(2) → (1): $2s^2 \cancel{+2} + 1 = 2s^3 \cancel{+3}$

$2s^2 - 2s^3 = 0$

$2s^2(s-1) = 0$

$s = 0 \quad$ or $\quad s = 1$

$\downarrow \qquad\qquad \downarrow$

$t = 1 \qquad\qquad t = 2$

check in (3): $1^2 - 1 \overset{?}{=} 3(0)$ ✓

$2^2 - 1 \overset{?}{=} 3(1)$ ✓

intersection

plug in: $\vec{r}_1(\overset{t}{1}) = \langle 3, 1, 0 \rangle$ and $\vec{r}_1(\overset{t}{2}) = \langle 5, 2, 3 \rangle$

(b) Determine the angle between the curves at the point P $(3, 1, 0)$.

↳ $t = 1$, $s = 0$

$\vec{r}_1{}'(t) = \langle 2, 1, 2t \rangle \to \vec{r}_1{}'(1) = \langle 2, 1, 2 \rangle$

$\vec{r}_2{}'(s) = \langle 6s^2, 2s, 3 \rangle \to \vec{r}_2{}'(0) = \langle 0, 0, 3 \rangle$

$\langle 2,1,2 \rangle \cdot \langle 0,0,3 \rangle = \| \langle 2,1,2 \rangle \| \, \| \langle 0,0,3 \rangle \| \cos\theta$

$6 = 9 \cos\theta$

$\frac{2}{3} = \cos\theta$

$\cos^{-1}\left(\frac{2}{3}\right) = \theta$

(c) Is there a plane containing both of the curves C_1 and C_2? If so, find a cartesian equation of the plane. If not, justify your answer.

would require common normal

$$\vec{n} = \vec{r}_1'(t) \times \vec{r}_2'(s)$$

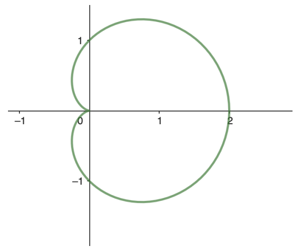
is constant (up to scalar).

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2t \\ 6s^2 & 2s & 3 \end{vmatrix} = \langle 3-4st,\ 12s^2t-6,\ 4s-6s^2 \rangle$$

$$\left.\begin{array}{ll} \text{test: } s=0, t=0: & \langle 3,-6,0\rangle \\ s=1, t=0: & \langle 3,-6,-2\rangle \end{array}\right] \text{ not } \parallel$$

so... not contained in a plane

Problem 5. ($\approx$ 20 points)



A *cardioid* C (pictured above) is a curve parameterized by

$$\mathbf{r}(t) = \left\langle \cos(t) + \frac{1}{2} + \frac{1}{2}\cos(2t),\ \sin(t) + \frac{1}{2}\sin(2t) \right\rangle, \text{ where } 0 \le t \le 2\pi.$$

(a) Use the above parametrization to set up an integral expressing the length of C. The expression of the integrand must be optimally simplified. You do NOT need to evaluate the integral.

$$\vec{r}\,'(t) = \langle -\sin t - \sin 2t\ ,\ \cos t + \cos 2t \rangle$$

$$\int_0^{2\pi} \|\vec{r}\,'(t)\|\, dt = \int_0^{2\pi} \sqrt{(\sin t + \sin 2t)^2 + (\cos t + \cos 2t)^2}\ dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + \sin^2 2t + \cos^2 2t + 2\sin t \sin 2t + 2\cos t\cos 2t}\ dt$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin t \sin 2t + 2\cos t \cos 2t}\ dt$$

(b) Another possible parametrization of the cardioid is given by

$$\mathbf{r}(t) = \left\langle \cos(t) + \cos^2(t), \sin(t) + \sin(t)\cos(t) \right\rangle, \text{ where } 0 \le t \le 2\pi.$$

Using the second parametrization, calculate the area of the region $\mathcal{D}$ enclosed by C.

$\vec{r}\,'(t) = \langle -\cos t - 2\sin t\cos t,\ \cos t + \cos^2 t - \sin^2 t \rangle$

Green's: $\langle P, Q \rangle = \langle 0, x \rangle \to Q_x - P_y = 1$

$\text{area} = \iint_D 1\, dA = \int_C 0\,dx + x\,dy$

$= \int_0^{2\pi} (\cos t + \cos^2 t)(\cos t + \cos^2 t - \sin^2 t)\, dt$

(int. to 0 by symmetry)

$= \int_0^{2\pi} \cos^2 t + \cancel{\cos^3 t} - \cancel{\cos t\sin^2 t} + \cancel{\cos^3 t} + \cos^4 t - \sin^2 t\cos^2 t\, dt$

$= \int_0^{2\pi} \frac{1}{2} + \underbrace{\cancel{\frac{1}{2}\cos 2\theta}}_{\text{int to } 0} + \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)^2 - \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt$

$= \int_0^{2\pi} \frac{1}{2} + \cancel{\frac{1}{4}} + \underbrace{\cancel{\frac{1}{2}\cos 2t}}_{\text{int to } 0} + \frac{1}{4}\cos^2 2t - \cancel{\frac{1}{4}} + \frac{1}{4}\cos^2 2t\, dt$

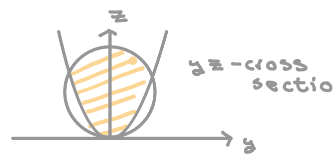
$= \int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos^2 2t\, dt$

$= \int_0^{2\pi} \frac{1}{2} + \frac{1}{4} + \underbrace{\cancel{\frac{1}{4}\cos 4t}}_{\text{int to } 0}\, dt$

$= \frac{6\pi}{4}$

Problem 6. ($\approx$ 18 points) Let $\mathcal{R}$ be the region enclosed by the sphere $x^2+y^2+z^2=4z$ and the paraboloid $z=x^2+y^2$.

(a) Set up an integral in spherical coordinates expressing the **volume** of $\mathcal{R}$. You do NOT need to evaluate the integral. Your work must be supported by properly-labelled sketch(es).



sphere: $x^2+y^2+z^2=4z \longrightarrow x^2+y^2+(z-2)^2=4$

$\rho^2 = 4\rho\cos\phi$

$\rho = 4\cos\phi$

paraboloid: $z = x^2+y^2 = r^2$

$\cancel{\rho}\cos\phi = \rho^{\cancel{2}}\sin^2\phi$

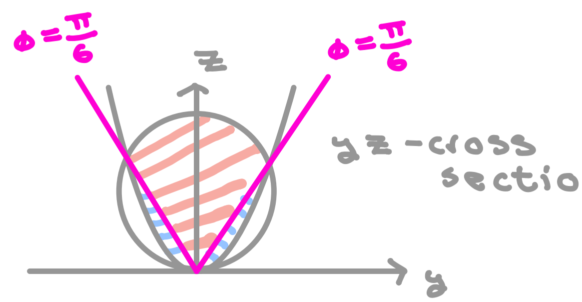
$\frac{\cos\phi}{\sin^2\phi} = \rho$

intersect: $4\cancel{\cos\phi} = \frac{\cancel{\cos\phi}^{\,1}}{\sin^2\phi}$

$4\sin^2\phi = 1$

$\sin\phi = \frac{1}{2}$

$\phi = \frac{\pi}{6}$



if $0 \le \phi \le \frac{\pi}{6}$: $0 \le \rho \le 4\cos\phi$

if $\frac{\pi}{6} \le \phi \le \frac{\pi}{2}$: $0 \le \rho \le \frac{\cos\phi}{\sin^2\phi}$

so: volume is

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{4\cos\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\cos\phi/\sin^2\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

(b) Let Ω be the surface contained on the sphere that lies inside the paraboloid. Express Ω as a parametric surface $\mathbf{r}(\theta, \Phi)$.

$\rho \sin\phi \cos\theta, \ \rho \sin\phi \sin\theta, \ \rho\cos\phi$

$$\vec{r}(\phi, \theta) = \langle 4\cos\phi \sin\phi \cos\theta, 4\cos\phi \sin\phi \sin\theta, 4\cos\phi\cos\phi \rangle$$

$$= \langle 2\sin 2\phi \cos\theta, 2\sin 2\phi \cos\theta, 2 + 2\cos 2\phi \rangle$$

w/ $0 \le \phi \le \frac{\pi}{6}, \ 0 \le \theta \le 2\pi$

(c) Set up an integral expressing the **surface area** of Ω. Do NOT evaluate the integral.

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4\cos 2\phi \cos\theta & 4\cos 2\phi \sin\theta & -4\sin 2\phi \\ -2\sin 2\phi \sin\theta & 2\sin 2\phi \cos\theta & 0 \end{vmatrix}$$

$$= \langle 8\sin^2 2\phi \cos\theta, 8\sin^2 2\phi \sin\theta, 8\sin 2\phi \cos 2\phi \rangle$$

$$= 8\sin 2\phi \langle \sin 2\phi \cos\theta, \sin 2\phi \sin\theta, \cos 2\phi \rangle$$

$$\|\vec{r}_\phi \times \vec{r}_\theta\| = 8\sin 2\phi \sqrt{1} = 8 \sin 2\phi$$

↳ ... simplify!

$$\text{area} = \int_0^{2\pi} \int_0^{\pi/6} 8 \sin 2\phi \, d\phi \, d\theta$$

topic (curvature) not covered

Problem 7. (≈ 20 points) Consider the curve C parametrized by

$$\mathbf{r}(t) = \left\langle 3 - \sin(2t),\, 2\cos(t),\, e^{\sin(t)} - 1 \right\rangle .$$

(a) Calculate the **TNB** frame of the curve C at the point $(3, -2, 0)$.

(b) ~~Calculate the tangential component of the acceleration vector at the point $(3, -2, 0)$.~~

(c) ~~Determine both the radius, and the coordinates of the center of the osculating circle to the curve C at the point $(3, -2, 0)$. You must carefully explain your methods and reasoning.~~

Problem 8. ($\approx$ 20 points) Consider the solid E bounded by the surfaces $z = x^2 + 1$, $z = y - 1$, and $x^2 + y^2 = 1$, with $x \geq 0$. Calculate the flux of the 3–dimensional vector field

$$\mathbf{F}(x, y, z) = \left\langle -5y^2 e^{-\arctan(z^2)},\, 4\ln(x^2 + y^2),\, \cos^2(x) e^{xy} \right\rangle$$

across the surface S bounding E, S being equipped with the inward orientation. You must carefully justify your methods; in particular, you must include relevant, properly-labelled sketch(es) supporting the logic of your reasoning.

$$\operatorname{div} \vec{F} = \frac{8y}{x^2+y^2}$$

E in cylindrical:

$x^2+1 \geq 1$

$y-1 \leq 0$

in region $x^2+y^2 \leq 1$

top: $z = x^2+1 \longrightarrow z = r^2\cos^2\theta + 1$

bottom: $z = y-1 \longrightarrow z = r\sin\theta - 1$

$x^2+y^2 \leq 1 \longrightarrow 0 \leq r \leq 1$

$x \geq 0 \rightarrow 0 \leq \theta \leq \frac{\pi}{2}$

$$\underset{\substack{S \\ \text{inward}}}{\iint} \vec{F}\cdot d\vec{S} = -\underset{E}{\iiint} \frac{8y}{x^2+y^2}\, dV$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_{r\sin\theta - 1}^{r^2\cos^2\theta + 1} \frac{8\cancel{r}\sin\theta}{\cancel{r^2}} \cdot \cancel{r}\, dz\, dr\, d\theta$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \cancelto{\text{int to } 0}{8r^2\sin\theta\cos^2\theta} + \cancelto{\text{int to } 0}{8\sin\theta} - 8r\sin^2\theta + \cancelto{\text{int to } 0}{8\sin\theta}\, dr\, d\theta$$

$$= + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 4\sin^2\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 - \cancelto{\text{int to } 0}{2\cos 2\theta}\, d\theta$$

$$= 2\pi$$

Problem 9. ($\approx$ 15 points) Evaluate the line integral

$$I = \int_C \left(y^2 + e^{x^2}\right) dx + (x + \sin(y))\, dy + (xy + \ln(z))\, dz,$$

where C is the curve parametrized by $\mathbf{r}(t) = \langle\, 2\sin(t), 2\cos(t), 2\sin(2t)\,\rangle$, $0 \le t \le 2\pi$. You must carefully justify your methods.

Hint: You may use the fact that the curve C lies on the surface S given by $z = xy$. To receive full credit, explain why this property holds.

$\vec{F} = \langle y^2 + e^{x^2}, x + \sin y, xy + \ln z \rangle$

note: $\mathcal{C}$ is a closed curve, so it bounds a surface

so: by Stokes, any $\vec{F}^{new}$ with same curl as $\vec{F}$ will have same integral

now: $\text{curl}\vec{F} = \langle x, -y, 1-2y \rangle$

but: $\vec{F}^{new} = \langle y^2, x, xy \rangle$ has the same curl

hence: $\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = \int_{\mathcal{C}} \vec{F}^{new} \cdot d\vec{s}$

$\vec{F}^{new}(\vec{r}(t)) = \langle 2\cos^2 t, 2\sin t, 4\sin t\cos t \rangle = \langle 2\cos^2 t, 2\sin t, 2\sin 2t \rangle$

$\vec{r}\,'(t) = \langle 2\cos t, -2\sin t, 4\cos 2t \rangle$

so: $\int_{\mathcal{C}} \vec{F}^{new} \cdot d\vec{s} = \int_0^{2\pi} \cancel{2\cos^3 t}^{\text{int to } 0} - 4\sin^2 t + \cancel{8\sin 2t\cos 2t}^{\text{int to } 0}\, dt$

$= \int_0^{2\pi} -2 + \cancel{2\sin 2t}^{\text{int to } 0}\, dt$

$= -4\pi$

ALT. APPROACH, NEXT PAGE

$$\vec{r}(t) = \langle \overset{x}{2\sin t}, \overset{y}{2\cos t}, \overset{z}{2\sin 2t} \rangle$$

Scratch, scratch zone...

Claim: $\mathcal{C}$ is on $z = xy$

Justification: sub in... $\cancel{2}\sin 2t \overset{?}{=} 2\sin t\, \cancel{2}\cos t$ ✓ double-angle fmla

next... $\mathcal{C}: \langle \underbrace{2\sin t, 2\cos t}_{x^2+y^2=4,\ \text{clockwise}}, xy \rangle$

so... $\mathcal{C}$ is boundary of $\mathcal{S}$ (downward) $: x^2+y^2 \leq 4,\ z = xy$

param. $\mathcal{S}$: $\vec{r}(x,y) = \langle x, y, xy \rangle,\ x^2+y^2 \leq 4$

$$\vec{r}_x \times \vec{r}_y = \langle -y, -x, 1 \rangle \xrightarrow{\text{downward}} \langle y, x, -1 \rangle$$

now... $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = \iint_{\mathcal{S}} \operatorname{curl}\vec{F} \cdot d\vec{S}$

$$= \iint_{x^2+y^2 \leq 4} \langle x, -y, 1-2y \rangle \cdot \langle y, x, -1 \rangle \, dA$$

$$= \iint_{x^2+y^2 \leq 4} \cancel{2y} - 1 \, dA$$

int to 0 by sym.

$$= -\text{area}(x^2+y^2 \leq 4)$$

$$= -4\pi$$

More scratch, scratch zone...